Chapter 9 • Curve Sketching

Review of Prerequisite Skills

2. c. $t^2 - 2t < 3$

$t^2 - 2t - 3 < 0$

$(t - 3)(t + 1) < 3$

Consider $t = 3$ and $t = -1$.

$> 0 \quad -1 \quad < 0 \quad 3 \quad > 0$

The solution is $-1 < t < 3$.

d. $x^2 + 3x - 4 > 0$

$(x + 4)(x - 1) > 0$

Consider $t = -4$ and $t = 1$.

$> 0 \quad -4 \quad < 0 \quad 1 \quad > 0$

The solution is $x < -4$ or $x > 1$.

4. b. $\lim\limits_{x\to 2} \dfrac{x^2 + 3x - 10}{x - 2}$

$= \lim\limits_{x\to 2} \dfrac{(x + 5)(x - 2)}{x - 2}$

$= \lim\limits_{x\to 2} (x + 5)$

$= 7$

Exercise 9.1

1. c. $f(x) = (2x - 1)^2(x^2 - 9)$

$f'(x) = 2(2x - 1)(2)(x^2 - 9) + 2x(2x - 1)^2$

Let $f'(x) = 0$:

$2(2x - 1)(2(x^2 - 9) + x(2x - 1)) = 0$

$2(2x - 1)(4x^2 - x - 18) = 0$

$2(2x - 1)(4x - 9)(x + 2) = 0$

$x = \dfrac{1}{2}$ or $x = \dfrac{9}{4}$ or $x = -2$.

The points are $\left(\dfrac{1}{2}, 0\right)$, (2.24, –48.2), and (–2, –125).

3. a. $y = x^7 - 430x^6 - 150x^3$

$\dfrac{dy}{dx} = 7x^6 - 2580x^5 - 450x^2$

If $x = 10$, $\dfrac{dy}{dx} < 0$.

If $x = 1000$, $\dfrac{dy}{dx} > 0$.

The curve rises upward in quadrant one.

c. $y = x\ln x - x^4$

$\dfrac{dy}{dx} = x\left(\dfrac{1}{x}\right) + \ln x - 4x^3$

$= 1 + \ln x - 4x^3$

If $x = 10$, $\dfrac{dy}{dx} < 0$.

If $x = 1000$, $\dfrac{dy}{dx} < 0$.

The curve is decreasing downward into quadrant four.

5. b. $f(x) = x^5 - 5x^4 + 100$

$f'(x) = 5x^4 - 20x^3$

Let $f'(x) = 0$:

$5x^4 - 20x^3 = 0$

$5x^3(x - 4) = 0$

$x = 0$ or $x = 4$.

x	$x < 0$	0	$0 < x < 4$	4	$x > 4$
$f'(x)$	+	0	–	0	+
Graph	Increasing		Decreasing		Increasing

d. $f(x) = \dfrac{x - 1}{x^2 + 3}$

$f'(x) = \dfrac{x^2 + 3 - 2x(x - 1)}{(x^2 + 3)^2}$

Let $f'(x) = 0$, therefore, $-x^2 + 2x + 3 = 0$.

Or $x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

$x = 3$ or $x = -1$

x	$x < -1$	–1	$-1 < x < 3$	3	$x > 3$
$f'(x)$	–	0	+	0	–
Graph	Decreasing		Increasing		Decreasing

e. $f(x) = x\ln(x)$

$$f'(x) = \ln x + \frac{1}{x}(x)$$
$$= \ln x + 1$$

Let $f'(x) = 0$:

$\ln x + 1 = 0$

$\ln x = -1$

$$x = e^{-1} = \frac{1}{e} = 0.37.$$

x	$x \le 0$	$0 < x < 0.37$	0.37	$x > 0.37$
$f'(x)$	No values	–	0	–
Graph		Decreasing		Increasing

6. $f'(x) = (x - 1)(x + 2)(x + 3)$

Let $f'(x) = 0$:

then $(x - 1)(x + 2)(x + 3) = 0$

$x = 1$ or $x = -2$ or $x = -3$.

x	$x < -3$	–3	$-3 < x < -3$	–2	$-2 < x < 1$	1	$x > 1$
$f'(x)$	–	0	+	0	–	0	+
Graph	Decreasing		Increasing		Decreasing		Increasing

7. $g'(x) = (3x - 2)\ln(2x^2 - 3x + 2)$

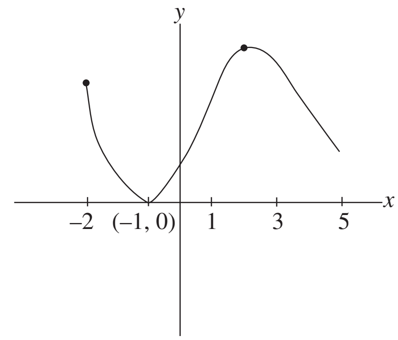
Let $g'(x) = 0$:

then $(3x - 2)\ln(2x^2 - 3x + 2) = 0$

$3x - 2 = 0$ or $\ln(2x^2 - 3x + 2) = 0$

$$x = \frac{2}{3} \text{ or } 2x^2 - 3x + 2 = e^0$$
$$2x^2 - 3x + 2 = 1$$
$$2x^2 - 3x + 1 = 0$$
$$(2x - 1)(x - 1) = 0$$
$$x = \frac{1}{2} \text{ or } x = 1.$$

x	$x < \frac{1}{2}$	$\frac{1}{2} < x < \frac{2}{3}$	$\frac{2}{3} < x < 1$	$x > 1$
$f'(x)$	–	+	–	+
Graph	Decreasing	Increasing	Decreasing	Increasing



9. $f(x) = x^3 + ax^2 + bx + c$

$f'(x) = 3x^2 + 2ax + b$

Since $f(x)$ increases to $(-3, 18)$ and then decreases, $f'(3) = 0$.

Therefore, $27 - 6a + b = 0$ or $6a - b = 27$. (1)

Since $f(x)$ decreases to the point $(1, -14)$ and then increases, $f'(1) = 0$.

Therefore, $3 + 2a + b = 0$ or $2a + b = -3$. (2)

Add (1) to (2): $8a = 24$ and $a = 3$.

When $a = 3$, $b = 6 + b = -3$ or $b = -9$.

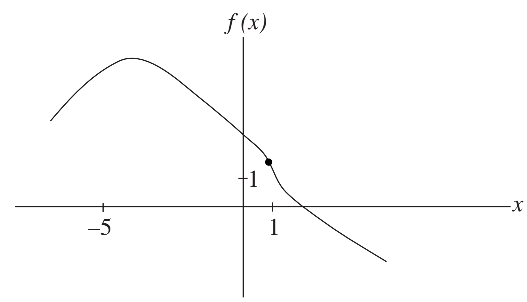
Since $(1, -14)$ is on the curve and $a = 3$, $b = -9$,

then $-14 = 1 + 3 - 9 + c$

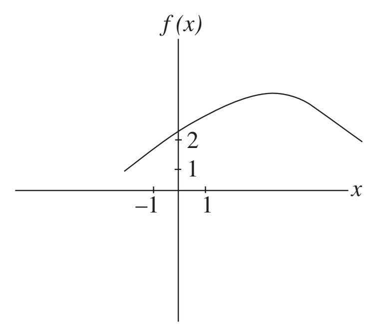
$c = -9$.

The function is $f(x) = x^3 + 3x^2 - 9x - 9$.

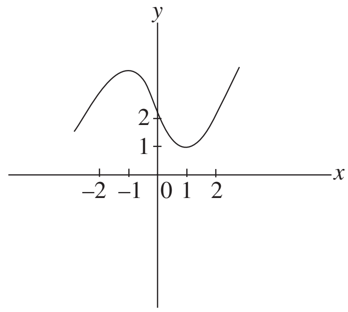
10.



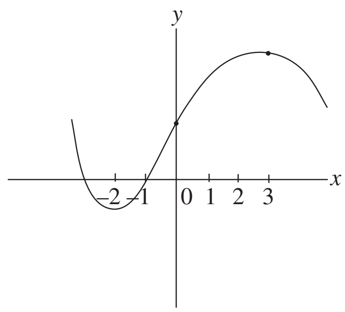
11. a.



b.



c.



12. $f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$

Let $f'(x) = 0$, then $x = \frac{-b}{2a}$.

If $x < \frac{-b}{2a}$, $f'(x) < 0$, therefore the function is decreasing.

If $x > \frac{-b}{2a}$, $f'(x) > 0$, therefore the function is increasing.

13. Let $y = f(x)$ and $u = g(x)$.

Let x_1 and x_2 be any two values in the interval $a \le x \le b$ so that $x_1 < x_2$.

Since $x_1 < x_2$, both functions are increasing:

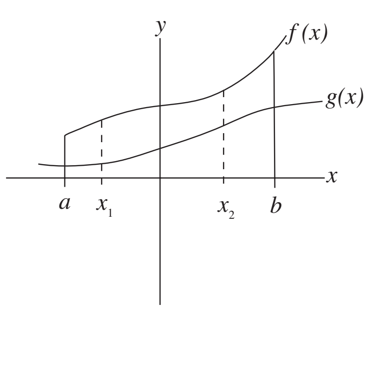
$f(x_2) > f(x_1)$ (1)

$g(x_2) > g(x_1)$ (2)

$yu = f(x) \bullet g(x)$.

(1) × (2) results in $f(x_2) \bullet g(x_2) > f(x_1)g(x_1)$.

The function yu or $f(x) \bullet g(x)$ is strictly increasing.



14. Let x_1, x_2 be in the interval $a \le x \le b$, such that $x_1 < x_2$.

Therefore, $f(x_2) > f(x_1)$, and $g(x_2) > g(x_1)$.

In this case, $f(x_1)$, $f(x_2)$, $g(x_1)$, and $g(x_2) < 0$.

Multiplying an inequality by a negative will reverse its sign.

Therefore, $f(x_2) \bullet g(x_2) < f(x_1) \bullet g(x_1)$.

But L.S. > 0 and R.S. > 0.

Therefore, the function fg is strictly increasing.

Exercise 9.2

3. b. $f(x) = \frac{2x}{x^2 + 9}$

$$f'(x) = \frac{2(x^2 + 9) - 2x(2x)}{(x^2 + 9)^2} = \frac{18 - 4x^2}{(x^2 + 9)^2}$$

Let $f'(x) = 0$:

therefore, $18 - 2x^2 = 0$

$x^2 = 9$

$x = \pm 3$.

x	$x < -3$	-3	$-3 < x < 3$	3	$x > 3$
$f'(x)$	–	0	+	0	–
Graph	Decreasing	Local Min	Increasing	Local Max	Decreasing

Local minimum at (–3, –0.3) and local maximum at (3, 0.3).

c. $y = xe^{-4x}$

$$\frac{dy}{dx} = e^{-4x} - 4xe^{-4x}$$

Let $\frac{dy}{dx} = 0$, $e^{-4x}(1 - 4x) = 0$:

$e^{-4x} \neq 0$ or $(1 - 4x) = 0$

$x = \frac{1}{4}$.

x	$x < \frac{1}{4}$	$\frac{1}{4}$	$x > \frac{1}{4}$
$f'(x)$	+	0	–
Graph	Increasing	Local Max	Decreasing

At $x = \frac{1}{4}$, $y = \frac{1}{4}e^{-1} = \frac{1}{4e}$.

Local maximum occurs at $\left(\frac{1}{4}, \frac{1}{4e}\right)$.

d. $y = \ln(x^2 - 3x + 4)$

$$\frac{dy}{dx} = \frac{2x - 3}{x^2 - 3x + 4}$$

Let $\frac{dy}{dx} = 0$, therefore, $2x - 3 = 0$

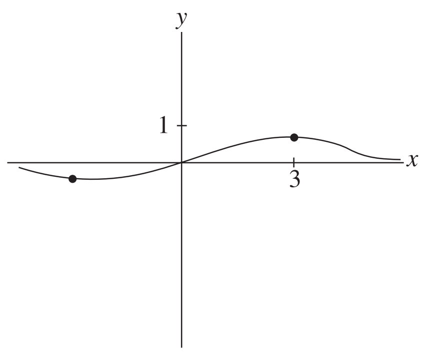
$$x = \frac{3}{2} = 1.5.$$

x	$x < 1.5$	1.5	$x > 1.5$
$f'(x)$	–	0	+
Graph	Decreasing	Local Min	Increasing

Local minimum at (1.5, ln 1.75).

4. b. $f(x) = \frac{2x}{x^2 + 9}$

The x-intercept is 0 and the y-intercept is 0.



c. $y = xe^{-4x}$

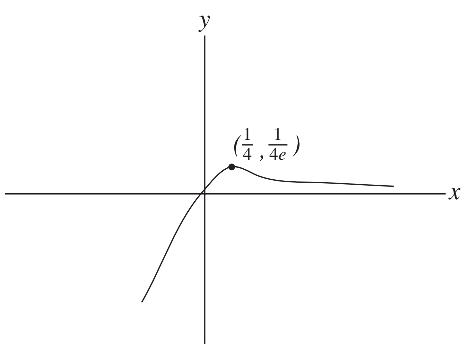
x-intercept, let $y = 0$,

$0 = xe^{-4x}$

Therefore, $x = 0$.

y-intercept, let $x = 0$,

$y = 0$.



d. $y = \ln(x^2 - 3x + 4)$

x-intercept, let $y = 0$,

$\ln x^2 - 3x + 4 = 0$

$x^2 - 3x + 4 = 1$

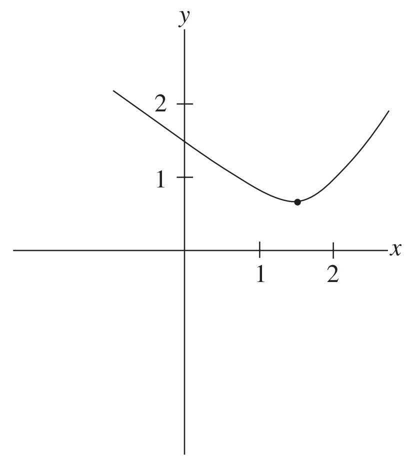
$x^2 - 3x + 3 = 0.$

No solution, since $0 = 9 - 12 < 0$

y-intercept, let $x = 0$,

$y = \ln 4$

$= 1.39.$



5. b. $s = -t^2e^{-3t}$

$$\frac{ds}{dt} = -2te^{-3t} + 3e^{-3t}(t^2)$$

Let $\frac{ds}{dt} = 0.$

$te^{-3t}[-2 + 3t] = 0$

$t = 0$ or $t = \frac{2}{3}$.

t	$t < 0$	$t = 0$	$0 < t < \frac{2}{3}$	$t = \frac{2}{3}$	$t > \frac{2}{3}$
$\frac{ds}{dt}$	+	0	–	0	+
Graph	Increasing	Local Max	Decreasing	Local Min	Increasing

Critical points are (0, 0) and $\left(\frac{2}{3}, -0.06\right)$.

Tangent is parallel to t-axis.

c. $y = (x-5)^{\frac{1}{3}}$

$$\frac{dy}{dx} = \frac{1}{3}\left(x-5\right)^{\frac{-2}{3}}$$

$$= \frac{1}{3(x-5)^{\frac{2}{3}}}$$

$$\frac{dy}{dx} \neq 0$$

The critical point is at (5, 0), but is neither a maximum or minimum. The tangent is not parallel to x-axis.

f. $y = x^2 - 12x^{\frac{1}{3}}$

$$\frac{dy}{dx} = 2x - \frac{1}{3}\left(12x^{\frac{-2}{3}}\right)$$

$$= 2x - \frac{4}{x^{\frac{2}{3}}}$$

Let $\frac{dy}{dx} = 0$. Then, $2x = \frac{4}{x^{\frac{2}{3}}}$:

$$2x^{\frac{5}{3}} = 4$$

$$x^{\frac{5}{3}} = 2$$

$$x = 2^{\frac{3}{5}} = \sqrt[5]{2^3}$$

$$x \doteq 1.5.$$

Critical points are at $x = 0$ and $x = 1.5$.

x	$x < 0$	$x = 0$	$0 < x < 1.5$	$x = 1.5$	$x > 1.5$
$\frac{dy}{dx}$	–	undefined	–	0	+
Graph	Decreasing	Vertical Tangent	Decreasing	Local Min	Increasing

Critical points are at (0, 0) and (1.5, –11.5).
Local minimum is at (1.5, –11.5).
Tangent is parallel to y-axis at (0, 0).
Tangent is parallel to x-axis at (1.5, –11.5).

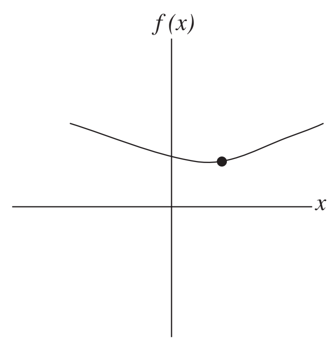
7. e. $f(x) = \sqrt{x^2 - 2x + 2}$

$$f'(x) = \frac{2x-2}{2\sqrt{x^2-2x+2}}$$

Let $f'(x) = 0$, then $x = 1$.
Also, $x^2 - 2x + 2 \geq 0$ for all x.

x	$x < 1$	$x = 1$	$x > 1$
$f'(x)$	–	0	+
Graph	Decreasing	Local Min	Increasing

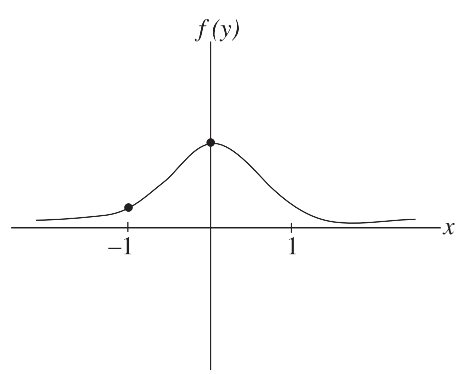
Local minimum is at (1, 1).



g. $f(x) = e^{-x^2}$
$f'(x) = -2xe^{-x^2}$
Let $f'(x) = 0$, then $x = 0$.

x	$x < 0$	0	$x > 0$
$f'(x)$	+	0	–
Graph	Increasing	Local Max	Decreasing

When $x = 0$, $e^0 = 1$.
Local maximum point is at (0, 1).



h. $f(x) = x^2 \ln x$

$$f'(x) = 2x \ln x + x^2\left(\frac{1}{x}\right)$$
$$= 2x \ln x + x$$

Let $f'(x) = 0$:

$2x \ln x + x = 0$

$x(2 \ln x + 1) = 0$

$x = 0$ or $\ln x = -\frac{1}{2}$.

But, $x > 0$, then $x \neq 0$,

$x = e^{-\frac{1}{2}} = 0.61$.

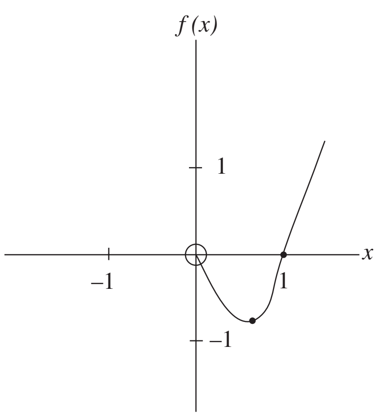
x	$0 < x < 0.61$	0.61	$x > 0.61$
$f'(x)$	–	0	+
Graph	Decreasing	Local Min	Increasing

Local minimum is at $x = 0.61$ and $f(0.61)$:

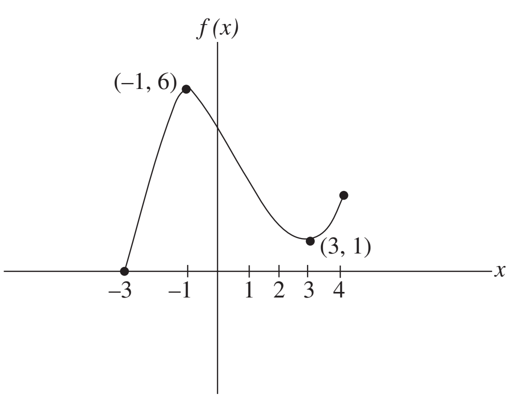
$= 2(0.61) \ln 0.61 + 0.61$

$= -0.64$.

Critical point is $(0.61, -0.64)$.



9.



10. $y = ax^2 + bx + c$

$$\frac{dy}{dx} = 2ax + b$$

Since a relative maximum occurs at $x = 3$, then $2ax + b = 0$ at $x = 3$.

Or, $6a + b = 0$.

Also, at $(0, 1)$, $1 = 0 + 0 + c$ or $c = 1$.

Therefore, $y = ax^2 + bx + 1$.

Since $(3, 12)$ lies on the curve,

$12 = 0a + 3b + 1$

$9a + 3b = 11$

$6a + b = 0$.

Since $b = -6a$,

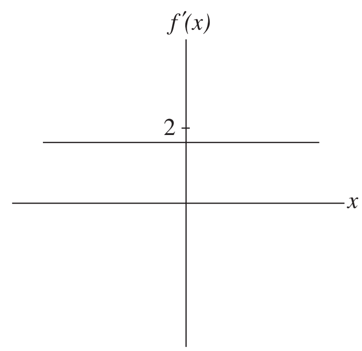
then $9a - 18a = 11$

or $a = \frac{-11}{9}$

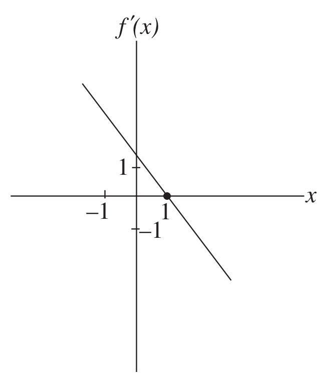
$b = \frac{22}{3}$.

The equation is $y = \frac{-11}{9}x^2 + \frac{22}{3}x + 1$.

11. a.



b.



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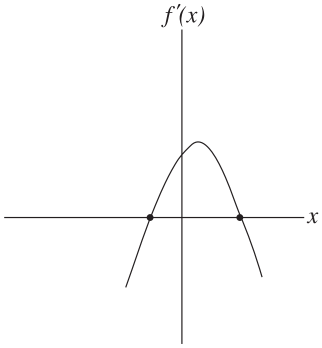
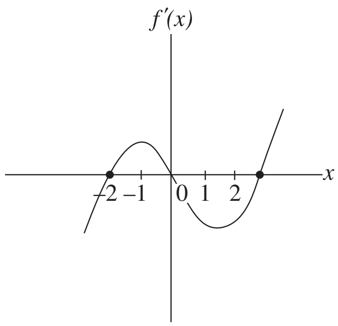


d.



12. $f(x) = 3x^4 + ax^3 + bx^2 + cx + d$

a. $f'(x) = 12x^3 + 3ax^2 + 2bx + c$

At $x = 0$, $f'(0) = 0$, then $f'(0) = 0 + 0 + 0 + c$

or $c = 0$.

At $x = -2$, $f'(-2) = 0$,

$-96 + 12a - 4b = 0$. (1)

Since $(0, -9)$ lies on the curve,

$-9 = 0 + 0 + 0 + 0 + d$

or $d = -9$.

Since $(-2, -73)$ lies on the curve,

$-73 = 48 - 8a + 4b + 0 - 9$

$-8a + 4b = -112$

or $2a - b = 28$ (2)

Also, from (1): $3a - b = 24$

$2a - b = -28$

$a = -4$

$b = -36$.

The function is $f(x) = 3x^4 - 4x^3 - 36x^2 - 9$.

b. $f'(x) = 12x^3 - 12x^2 - 72x$

Let $f'(x) = 0$:

$x^3 - x^2 - 6x = 0$

$x(x - 3)(x + 2) = 0$.

Third point occurs at $x = 3$,

$f(3) = -198$.

x	$x < -2$	-2	$-2 < x < 0$	0	$0 < x < 3$	3	$x > 3$
$f'(x)$	–	0	+	0	–	0	+
Graph	Decreasing	Local Min	Increasing	Local Max	Decreasing	Local Min	Increasing

Local minimum is at $(-2, -73)$ and $(3, -198)$.

Local maximum is at $(0, -9)$.

13. a. $y = 4 - 3x^2 - x^4$

$$\frac{dy}{dx} = -6x - 4x^3$$

Let $\frac{dy}{dx} = 0$:

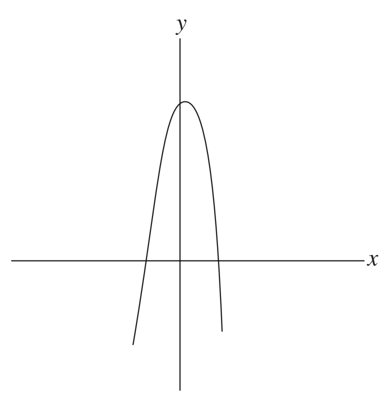
$-6x - 4x^3 = 0$

$-2x(2x^2 + 3) = 0$

$x = 0$ or $x^2 = \frac{-3}{2}$; inadmissible.

x	$x < 0$	0	$x > 0$
$\frac{dy}{dx}$	+	0	–
Graph	Increasing	Local Max	Decreasing

Local maximum is at $(0, 4)$.



b. $y = 3x^5 - 5x^3 - 30x$

$$\frac{dy}{dx} = 15x^4 - 15x^2 - 30$$

Let $\frac{dy}{dx} = 0$:

$15x^4 - 15x^2 - 30 = 0$

$x^4 - x^2 - 2 = 0$

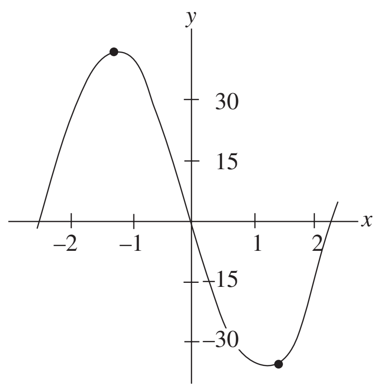
$(x^2 - 2)(x^2 + 1) = 0$

$x^2 = 2$ or $x^2 = -1$

$x = \pm\sqrt{2}$; inadmissible.

At $x = 100$, $\dfrac{dy}{dx} > 0$.

Therefore, function is increasing into quadrant one, local minimum is at (1.41, –39.6) and local maximum is at (–1.41, 39.6).



14. $h(x) = \dfrac{f(x)}{g(x)}$

Since $f(x)$ has a local maximum at $x = c$, then $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$.
Since $g(x)$ has a local minimum at $x = c$, then $g'(x) < 0$ for $x < c$ and $g'(x) > 0$ for $x > c$.

$$h(x) = \frac{f(x)}{g(x)}$$

$$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$$

If $x < c$, $f'(x) > 0$ and $g'(x) < 0$, then $h'(x) > 0$.
If $x > c$, $f'(x) < 0$ and $g'(x) > 0$, then $h'(x) < 0$.
Since for $x < c$, $h'(x) > 0$ and for $x > c$, $h'(x) < 0$.
Therefore, $h(x)$ has a local maximum at $x = c$.

Exercise 9.3

2. $f(x) = \dfrac{g(x)}{h(x)}$

Conditions for a vertical asymptote:
$h(x) = 0$ must have at least one solution s, and $\lim\limits_{x \to x_1} f(x) = \infty$.

Conditions for a horizontal asymptote:
$\lim\limits_{x \to \infty} f(x) = k$, where $k \in R$,

or $\lim\limits_{x \to -\infty} f(x) = k$ where $k \in R$.

Condition for an oblique asymptote is that the highest power of $g(x)$ must be one more than the highest power of $k(x)$.

6. **a.** $y = \dfrac{x - 3}{x + 5}$

$$\lim_{x \to -5^+} \frac{x-3}{x+5} = -\infty, \lim_{x \to -5^-} \frac{x-3}{x+5} = +\infty \quad (1)$$

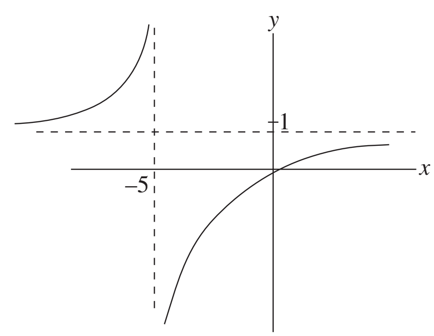
Vertical asymptote at $x = -5$.

$$\lim_{x \to \infty} \frac{x-3}{x+5} = 1, \lim_{x \to -\infty} \frac{x-3}{x+5} = 1 \quad (2)$$

Horizontal asymptote at $y = 1$.

$$\frac{dy}{dx} = \frac{x + 5 - x + 3}{(x+5)^2} = \frac{8}{(x+5)^2} \quad (3)$$

Since $\dfrac{dy}{dx} \neq 0$, there are no maximum or minimum points.



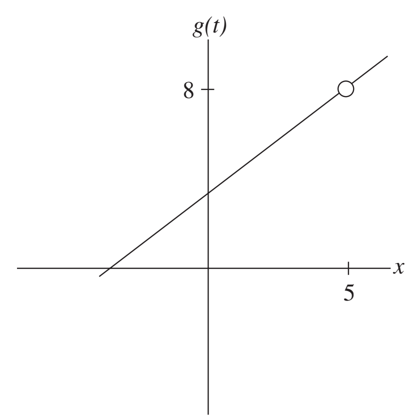
c. $g(t) = \dfrac{t^2 - 2t - 15}{t - 5}$

Discontinuity at $t = 5$.

$$\lim_{t \to 5^-} \frac{(t-5)(t+3)}{t-5} = \lim_{t \to 5^-} (t+2) = 8$$

$$\lim_{t \to 5^+} (t+3) = 8$$

No asymptote at $x = 5$. The curve is of the form $t + 3$.



d. $p(x) = \dfrac{15}{6 - 2e^x}$

Discontinuity when $6 - 2e^x = 0$

$$e^x = 3$$
$$x = \ln 3 \doteq 1.1.$$

$\lim\limits_{x\to 1.1^-} \dfrac{15}{6-2e^x} = +\infty$, $\lim\limits_{x\to 1.1^+} \dfrac{15}{6-2e^x} = +\infty$ (1)

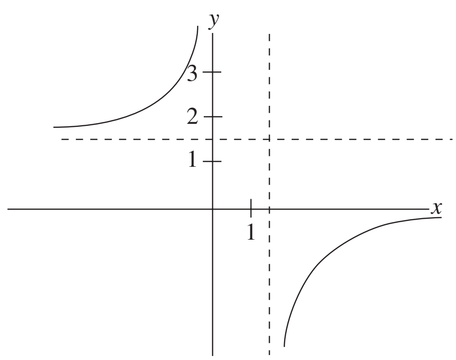
Vertical asymptote at $x \doteq 1.1$.

Horizontal asymptote: $\lim\limits_{x\to\infty} \dfrac{15}{6-2e^x} = 0$ from below, (2)

$\lim\limits_{x\to\infty} \dfrac{15}{6-2e^x} = \dfrac{15}{6}$ from above.

$p'(x) = \dfrac{-15(-2e^x)}{(6-2e^x)^2}$ (3)

True if $e^x = 0$, which is not possible. No maximum or minimum points.



e. $y = \dfrac{(2+x)(3-2x)}{x^2 - 3x}$

Discontinuity at $x = 0$ and $x = 3$ (1)

$\lim\limits_{x\to 0^+} \dfrac{(2+x)(3-2x)}{x^2-3x} = +\infty$

$\lim\limits_{x\to 0^-} \dfrac{(2+x)(3-2x)}{x^2-3x} = -\infty$

$\lim\limits_{x\to 3^+} \dfrac{(2+x)(3-2x)}{x^2-3x} = +\infty$

$\lim\limits_{x\to 3^-} \dfrac{(2+x)(3-2x)}{x^2-3x} = -\infty$

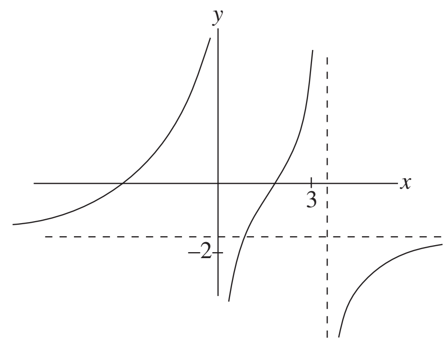
Vertical asymptotes at $x = 0$ and $x = 3$.

Horizontal asymptote. (2)

$$\lim_{x\to\infty} \frac{(2+x)(3-2x)}{x^2-3x} = \lim_{x\to\infty} \frac{\frac{6}{x^2} - \frac{1}{x} - 2}{1 - \frac{3}{x}} = -2$$

$$\lim_{x\to-\infty} \frac{\frac{6}{x^2} - \frac{1}{x} - 2}{1 - \frac{3}{x}} = -2$$

Horizontal asymptote at $y = -2$.



f. $P = \dfrac{10}{n^2 + 4}$

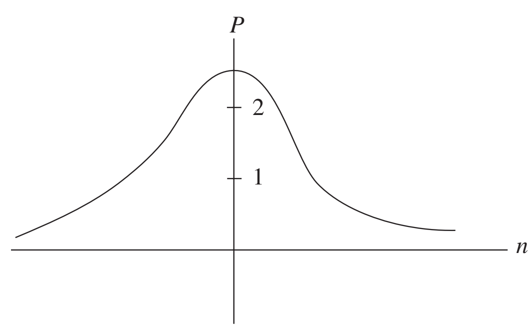
No discontinuity

$\lim\limits_{n\to\infty} p = 0$, $\lim\limits_{n\to\infty} p = 0$

$$\frac{dp}{dn} = \frac{-10(2n)}{(n^2+4)^2}$$

$\dfrac{dp}{dn} = 0$, then $n = 0$

Maximum point is at (0, 2.5).



7. **b.** $f(x) = \dfrac{2x^2 + 9x + 2}{2x + 3}$

$$\begin{array}{r|l} & \phantom{2x^2+{}} x + 3 \\ \hline 2x + 3 & 2x^2 + 9x + 2 \\ & \underline{2x^2 + 3x} \\ & \phantom{2x^2+{}} 6x + 2 \\ & \phantom{2x^2+{}} \underline{6x + 9} \\ & \phantom{2x^2+6x+{}} -7 \end{array}$$

$f(x) = \dfrac{2x^2 + 9x + 2}{2x + 3}$

$= x + 3 - \dfrac{7}{2x + 3}$

Oblique asymptote is at $y = x + 3$.

d. $f(x) = \dfrac{x^3 - x^2 - 9x + 15}{x^2 - 4x + 3}$

$$\begin{array}{r|l} & \phantom{x^3 - x^2 - 9x +{}} x + 3 \\ \hline x^2 - 4x + 3 & x^3 - x^2 - 9x + 15 \\ & \underline{x^3 - 4x^2 + 3x} \\ & \phantom{x^3 - {}} 3x^2 - 12x + 15 \\ & \phantom{x^3 - {}} \underline{3x^2 - 12x + 9} \\ & \phantom{x^3 - 3x^2 - 12x + {}} 6 \end{array}$$

$f(x) = x + 3 + \dfrac{6}{x^2 - 4x + 3}$

Oblique asymptote is at $y = x + 3$.

8. **b.** Oblique asymptote is at $y = x + 3$.

Consider $x > \dfrac{-3}{2}$ and $x < \dfrac{-3}{2}$.

Consider $x = 0$.

$f(0) = \dfrac{2}{3}$ and for the oblique asymptote $y = 3$.

Therefore, the oblique asymptote is above the

curve for $x > \dfrac{-3}{2}$.

The curve approaches the asymptote from below.
Consider $x = -2$.

$f(-2) = \dfrac{8 - 18 + 2}{-1}$

$= 8$

For the oblique asymptote, $y = 1$.
Therefore, the curve is above the oblique asymptote and approaches the asymptote from above.

9. **a.** $f(x) = \dfrac{3 - x}{2x + 5}$

Discontinuity is at $x = -2.5$.

$\lim\limits_{x \to -2.5^-} \dfrac{3 - x}{2x + 5} = -\infty$

$\lim\limits_{x \to -2.5^+} \dfrac{3 - x}{2x + 5} = +\infty$

Vertical asymptote is at $x = -2.5$.

Horizontal asymptote:

$\lim\limits_{x \to \infty} \dfrac{3 - x}{2x + 5} = -\dfrac{1}{2}$, $\lim\limits_{x \to -\infty} \dfrac{3 - x}{2x + 5} = -\dfrac{1}{2}$.

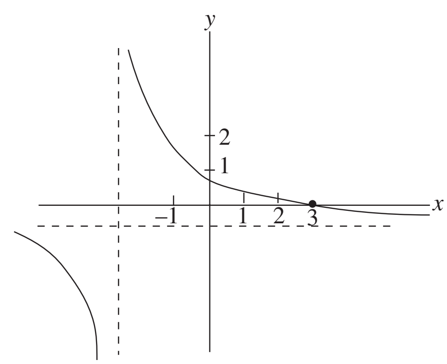
Horizontal asymptote is at $y = -\dfrac{1}{2}$.

$f'(x) = \dfrac{-(2x + 5) - 2(3 - x)}{(2x + 5)^2} = \dfrac{-11}{(2x + 5)^2}$

Since $f'(x) \neq 0$, there are no maximum or minimum points.

y-intercept, let $x = 0$, $y = \dfrac{3}{5} = 0.6$

x-intercept, let $y = 0$, $\dfrac{3 - x}{2x + 5} = 0$, $x = 3$



d. $s(t) = t + \dfrac{1}{t}$

Discontinuity is at $t = 0$.

$\lim\limits_{t \to 0^+} \left(t + \dfrac{1}{t}\right) = +\infty$

$\lim\limits_{t \to 0^-} \left(t + \dfrac{1}{t}\right) = -\infty$

Oblique asymptote is at $s(t) = t$.

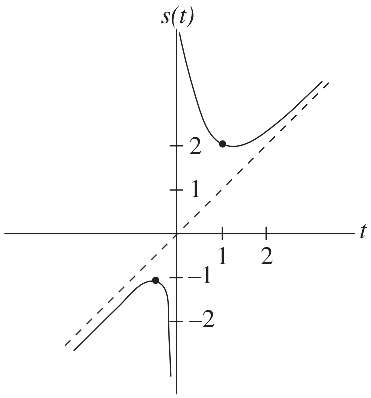
$s'(t) = 1 - \dfrac{1}{t^2}$

Let $s'(t) = 0$, $t^2 = 1$

$t = \pm 1$.

t	$t < -1$	$t = -1$	$-1 < t < 0$	$0 < t < 1$	$t = 1$	$t > 1$
$s'(t)$	+	0	–	–	0	+
Graph	Increasing	Local Max	Decreasing	Decreasing	Local Min	Increasing

Local maximum is at (–1, –2) and local minimum is at (1, 2).



e. $g(x) = \dfrac{2x^2 + 5x + 2}{x + 3}$

Discontinuity is at $x = -3$.

$$\frac{2x^2 + 5x + 2}{x + 3} = 2x - 1 + \frac{5}{x + 3}$$

Oblique asymptote is at $y = 2x - 1$.

$\lim\limits_{x \to -3^+} g(x) = +\infty$, $\lim\limits_{x \to -3^-} g(x) = -\infty$

$$g'(x) = \frac{(4x + 5)(x + 3) - (2x^2 + 5x + 2)}{(x + 3)^2}$$

$$= \frac{2x^2 + 12x + 13}{(x + 3)^2}$$

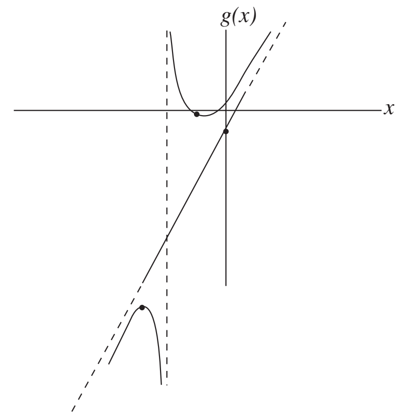
Let $g'(x) = 0$, therefore, $2x^2 + 12x + 13 = 0$:

$$x = \frac{-12 \pm \sqrt{144 - 104}}{4}$$

$x = -1.4$ or $x = -4.6$.

x	$x < -4.6$	-4.6	$-4.6 < x < -3$	-3	$-3 < x < -1.4$	$x = 1.4$	$x > -1.4$
$g'(x)$	+	0	–	Undefined	–	0	+
Graph	Increasing	Local Max	Decreasing	Vertical Asymptote	Decreasing	Local Min	Increasing

Local maximum is at (–4.6, –10.9) and local minimum is at (–1.4, –0.7).



f. $s(t) = \dfrac{t^2 + 4t - 21}{t - 3}$, $t \geq -7$

$$= \frac{(t + 7)(t - 3)}{(t - 3)}$$

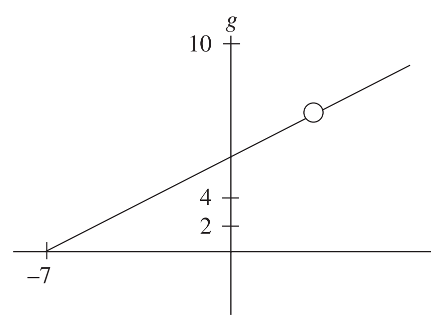
Discontinuity is at $t = 3$.

$$\lim_{x \to 3^+} \frac{(t + 7)(t - 3)}{t - 3)} = \lim_{x \to 3^+} (t + 7)$$

$$= 10$$

$$\lim_{x \to 3^-} (t + 7) = 10$$

There is no vertical asymptote.
The function is the straight line $s = t + 7$, $t \geq -7$.



11. $f(x) = \dfrac{ax + 5}{3 - bx}$

Vertical asymptote is at $x = -4$.
Therefore, $3 - bx = 0$ at $x = -5$.
That is, $3 - b(-5) = 0$

$$b = \frac{3}{5}.$$

Horizontal asymptote is at $y = -3$.

$$\lim_{x \to \infty} \left(\frac{ax + 5}{3 - bx}\right) = -3$$

$$\lim_{x\to\infty}\left(\frac{ax+5}{3-bx}\right)=\lim_{x\to\infty}\left(\frac{a+\frac{5}{x}}{\frac{3}{x}-b}\right)=\frac{-a}{b}$$

But $-\frac{a}{b}=-3$ or $a=3b$.

But $b=\frac{3}{5}$, then $a=\frac{9}{5}$.

12. a. $$\lim_{x\to\infty}\frac{x^2+1}{x+1}=\lim_{x\to\infty}\frac{x+\frac{1}{x}}{1+\frac{1}{x}}$$
$$=\infty$$

$$\lim_{x\to\infty}\frac{x^2+2x+1}{x+1}=\lim_{x\to\infty}\frac{(x+1)(x+1)}{(x+1)}$$
$$=\lim_{x\to\infty}(x+1)$$
$$=\infty$$

b. $$\lim_{x\to\infty}\left[\frac{x^2+1}{x+1}-\frac{x^2+2x+1}{x+1}\right]$$
$$=\lim_{x\to\infty}\frac{x^2+1-x^2-2x-1}{x+1}$$
$$=\lim_{x\to\infty}\frac{-2x}{x+1}$$
$$=\lim_{x\to\infty}\frac{-2}{1+\frac{1}{x}}=-2$$

13. $f(x)=\dfrac{2x^2-2x}{x^2-9}$

Discontinuity is at $x^2-9=0$ or $x=\pm3$.

$\lim_{x\to3^+}f(x)=+\infty$

$\lim_{x\to3^-}f(x)=-\infty$

$\lim_{x\to-3^+}f(x)=-\infty$

$\lim_{x\to-3^-}f(x)=+\infty$

Vertical asymptotes are at $x=3$ and $x=-3$.

Horizontal asymptote:

$\lim_{x\to\infty}f(x)=2$ (from below)

$\lim_{x\to\infty}f(x)=2$ (from above)

Horizontal asymptote is at $y=2$.

$$f'(x)=\frac{(4x-2)(x^2-9)-2x(2x^2-2x)}{(x^2-9)^2}$$
$$=\frac{4x^3-2x^2-36x+18-4x^3+4x^2}{(x^2-9)^2}$$
$$=\frac{2x^2-36x+18}{(x^2-9)^2}$$

Let $f'(x)=0$, $2x^2-36x+18=0$
or $x^2-18x+9=0$.

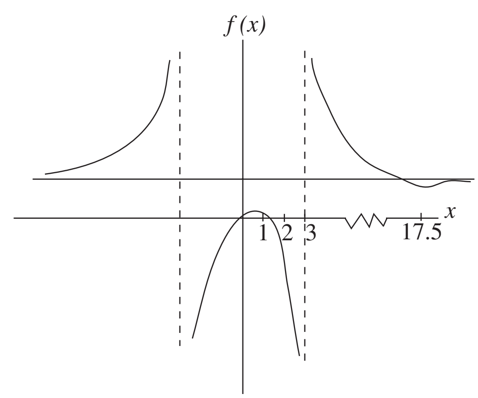
$$x=\frac{18\pm\sqrt{18^2-36}}{2}$$

$x=0.51$ or $x=17.5$

$y=0.057$ or $y=1.83$.

x	$-3<x<0.51$	0.51	$0.51<x<3$	$3<x<17.5$	17.5	$x>17.5$
$f'(x)$	+	0	–	–	0	+
Graph	Increasing	Local Max	Decreasing	Decreasing	Local Min	Increasing

Local maximum is at (0.51, 0.057) and local minimum is at (17.5, 1.83).



14. $y=\dfrac{x^2+3x+7}{x+2}$

$$\begin{array}{r|l} & x+1 \\ x+2 & x^2+3x+7 \\ & \underline{x^2+2x} \\ & \quad\; x+7 \\ & \quad\; \underline{x+2} \\ & \qquad 5 \end{array}$$

$$y=\frac{x^2+3x+7}{x+2}=x+1+\frac{5}{x+2}$$

Oblique asymptote is at $y=x+1$.

Exercise 9.4

2. a. $y = x^3 - 6x^2 - 15x + 10$

$\frac{dy}{dx} = 3x^2 - 12x - 15$

For critical values, we solve $\frac{dy}{dx} = 0$:

$3x^2 - 12x - 15 = 0$

$x^2 - 4x - 5 = 0$

$(x - 5)(x + 1) = 0$

$x = 5$ or $x = -1$

The critical points are (5, –105) and (–1, 20).

Now, $\frac{d^2y}{dx^2} = 6x - 12$.

At $x = 5$, $\frac{d^2y}{dx^2} = 18 > 0$. There is a local minimum at this point.

At $x = -1$, $\frac{d^2y}{dx^2} = -18 < 0$. There is a local maximum at this point.

The local minimum is (5, –105) and the local maximum is (–1, 20)

b. $y = \frac{25}{x^2 + 48}$

$\frac{dy}{dx} = -\frac{50x}{(x^2 + 48)^2}$

For critical values, solve $\frac{dy}{dx} = 0$ or $\frac{dy}{dx}$ does not exist.

Since $x^2 + 48 > 0$ for all x, the only critical point is $\left(0, \frac{25}{48}\right)$.

$\frac{d^2y}{dx^2} = -50(x^2 + 48)^{-2} + 100x(x^2 + 48)^{-3}\,(2x)$

$= -\frac{50}{(x^2 + 48)^2} + \frac{200x^2}{(x^2 + 48)^3}$

At $x = 0$, $\frac{d^2y}{dx^2} = -\frac{50}{48^2} < 0$. The point $\left(0, \frac{25}{48}\right)$ is a local maximum.

c. $s = t + t^{-1}$

$\frac{ds}{dt} = 1 - \frac{1}{t^2}, t \neq 0$

For critical values, we solve $\frac{ds}{dt} = 0$:

$1 - \frac{1}{t^2} = 0$

$t^2 = 1$

$t = \pm 1$.

The critical points are (–1, –2) and (1, 2).

$\frac{d^2s}{dt^2} = \frac{2}{t^3}$

At $t = -1$, $\frac{d^2s}{dt} = -2 < 0$. The point (–1, –2) is a local maximum. At $t = 1$, $\frac{d^2s}{dt^2} = 2 > 0$. The point (1, 2) is a local minimum.

d. $y = (x - 3)^3 + 8$

$\frac{dy}{dx} = 3(x - 3)^2$

$x = 3$ is a critical value.

The critical point is (3, 8).

$\frac{d^2y}{dx^2} = 6(x - 3)$

At $x = 3$, $\frac{d^2y}{dx^2} = 0$.

The point (3, 8) is neither a relative (local) maximum or minimum.

3. **a.** For possible point(s) of inflection, solve

$$\frac{d^2y}{dx^2} = 0:$$

$$6x - 8 = 0$$

$$x = \frac{4}{3}.$$

Interval	$x < \frac{4}{3}$	$x = \frac{4}{3}$	$x > \frac{4}{3}$
$f''(x)$	< 0	$= 0$	> 0
Graph of of $f(x)$	Concave Down	Point of Inflection	Concave Up

The point $\left(\frac{4}{3}, -14\frac{20}{27}\right)$ is a point of inflection.

b. For possible point(s) of inflection, solve

$$\frac{d^2y}{dx^2} = 0:$$

$$\frac{200x^2 - 50x^2 - 2400}{(x^2 + 48)^3} = 0$$

$150x^2 = 2400.$

Since $x^2 + 48 > 0$:

$x = \pm 4.$

Interval	$x < -4$	$x = -4$	$-4 < x < 4$	$x = 4$	$x > 4$
$f''(x)$	> 0	$= 0$	< 0	$= 0$	> 0
Graph of $f(x)$	Concave Up	Point of Inflection	Concave Down	Point of Inflection	Concave Up

$\left(-4, \frac{25}{64}\right)$ and $\left(4, \frac{25}{64}\right)$ are points of inflection.

c. $\frac{d^2s}{dt^2} = \frac{3}{t^2}$

Interval	$t < 0$	$t = 0$	$t > 0$
$f''(t)$	< 0	Undefined	> 0
Graph of $f(t)$	Concave Down	Undefined	Concave Up

The graph does not have any points of inflection.

d. For possible points of inflection, solve

$$\frac{d^2y}{dx^2} = 0:$$

$$6(x - 3) = 0$$

$$x = 3.$$

Interval	$x < 3$	$x = 3$	$x > 3$
$f''(x)$	< 0	$= 0$	> 0
Graph of $f(x)$	Concave Down	Point of Inflection	Concave Up

(3, 8) is a point of inflection.

4. **a.** $f(x) = 2x^3 - 10x + 3$ at $x = 2$

$f'(x) = 6x^2 - 10$

$f''(x) = 12x$

$f''(2) = 24 > 0$

The curve lies above the tangent at (2, –1).

b. $g(x) = x^2 - \frac{1}{x}$ at $x = -1$

$g'(x) = 2x + \frac{1}{x^2}$

$g''(x) = 2 - \frac{2}{x^3}$

$g''(-1) = 2 + 2 = 4 > 0$

The curve lies above the tangent line at (–1, 2).

c. $s = e^t \ln t$ at $t = 1$

$$\frac{ds}{dt} = e^t \ln t + \frac{e^t}{t}$$

$$\frac{d^2s}{dt^2} = e^t \ln t + \frac{e^t}{t} + \frac{e^t}{t} - \frac{e^t}{t^2}$$

At $t = 1$, $\frac{d^2s}{dt^2} = 0 + e + e - e = e > 0.$

The curve is above the tangent line at (1, 0).

d. $p = \frac{w}{\sqrt{w^2 + 1}}$ at $w = 3$

$$p = w(w^2 + 1)^{-\frac{1}{2}}$$

$$\frac{dp}{dw} = (w^2 + 1)^{-\frac{1}{2}} + w\left(-\frac{1}{2}\right)(w^2 + 1)^{-\frac{3}{2}}(2w)$$

$$= (w^2 + 1)^{-\frac{1}{2}} - w^2(w^2 + 1)^{-\frac{3}{2}}$$

$$\frac{d^2p}{dw^2} = -\frac{1}{2}(w^2+1)^{-\frac{3}{2}}(2w) - 2w(w^2+1)^{-\frac{3}{2}} + w^2\left(\frac{3}{2}\right)(w^2+1)^{-\frac{5}{2}}\left(2w\right)$$

At $w = 3$, $\frac{d^2p}{dw^2} = -\frac{3}{10\sqrt{10}} - \frac{6}{10\sqrt{10}} + \frac{81}{100\sqrt{10}}$

$$= -\frac{9}{100\sqrt{10}} < 0.$$

The curve is below the tangent line at $\left(3, \frac{3}{\sqrt{10}}\right)$.

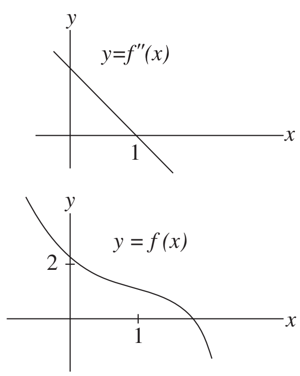
5. **(i) a.** $f''(x) > 0$ for $x < 1$

Thus, the graph of $f(x)$ is concave up on $x < 1$.

$f''(x) \leq 0$ for $x > 1$. The graph of $f(x)$ is concave down on $x > 1$.

(i) b. There is a point of inflection at $x = 1$.

(i) c.



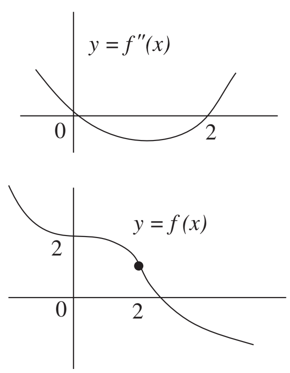
(ii) a. $f''(x) > 0$ for $x < 0$ or $x > 2$

The graph of $f(x)$ is concave up on $x < 0$ or $x > 2$.

The graph of $f(x)$ is concave down on $0 < x < 2$.

(ii) b. There are points of inflection at $x = 0$ and $x = 2$.

(ii) c.



6. For any function $y = f(x)$, find the critical points, i.e., the values of x such that $f'(x) = 0$ or $f'(x)$ does not exist. Evaluate $f''(x)$ for each critical value. If the value of the second derivative at a critical point is positive, the point is a local minimum. If the value of the second derivative at a critical point is negative, the point is a local maximum.

7. Step 4: Use the first derivative test or the second derivative test to determine the type of critical points that may be present.

8. **a.** $f(x) = x^4 + 4x^3$

(i) $f'(x) = 4x^3 + 12x^2$

$f''(x) = 12x^2 + 24x$

For possible points of inflection, solve $f''(x) = 0$:

$12x^2 + 24x = 0$

$12x(x + 2) = 0$

$x = 0$ or $x = -2$.

Interval	$x < -2$	$x = -2$	$-2 < x < 0$	$x = 0$	$x > 0$
$f''(x)$	> 0	$= 0$	< 0	$= 0$	> 0
Graph of $f(x)$	Concave Up	Point of Inflection	Concave Down	Point of Inflection	Concave Up

The points of inflection are $(-2, -16)$ and $(0, 0)$.

(ii) If $x = 0$, $y = 0$.

For critical points, we solve $f'(x) = 0$:

$4x^3 + 12x^2 = 0$

$4x^2(x + 3) = 0$

$x = 0$ or $x = -3$.

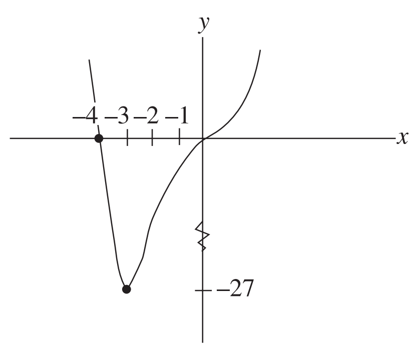
Interval	$x < -3$	$x = -3$	$-3 < x < 0$	$x = 0$	$x > 0$
$f'(x)$	< 0	0	> 0	$= 0$	> 0
Graph of $f(x)$	Decreasing	Local Min	Increasing		Increasing

If $y = 0$, $x^4 + 4x^3 = 0$

$x^3(x + 4) = 0$

$x = 0$ or $x = -4$.

The x-intercepts are 0 and -4.



b. $y = x - \ln x$

(i) $\frac{dy}{dx} = 1 - \frac{1}{x}$

$\frac{d^2y}{dx^2} = \frac{1}{x^2}$

Since $x > 0$, $\frac{d^2y}{dx^2} > 0$ for all x. The graph of $y = f(x)$ is concave up throughout the domain.

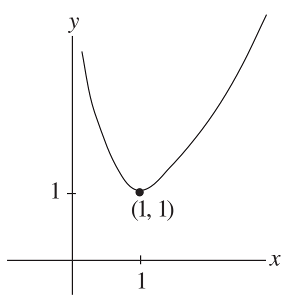
(ii) There are no x– or y-intercepts ($x > \ln x$ for all $x > 0$).

For critical points, we solve $\frac{dy}{dx} = 0$:

$1 - \frac{1}{x} = 0$

$x = 1.$

Interval	$0 < x < 1$	$x = 1$	$x > 1$
$\frac{dy}{dx}$	< 0	$= 0$	> 0
Graph of $y = f(x)$	Decreasing	Local Min	Increasing



c. $y = e^x + e^{-x}$

(i) $\frac{dy}{dx} = e^x - e^{-x}$

$\frac{d^2y}{dx^2} = e^x + e^{-x} > 0$, since $e^x > 0$ and $e^{-x} > 0$ for all x.

The graph of $y = f(x)$ is always concave up.

(ii) For critical points, we solve $\frac{dy}{dx} = 0$:

$e^x - e^{-x} = 0$

$e^x = \frac{1}{e^x}$

$(e^x)^2 = 1$

$e^x = 1$, since $e^x > 0$

$x = 0.$

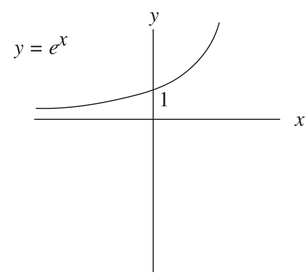
There are no x-intercepts ($e^x + e^{-x} > 0$ for all x).

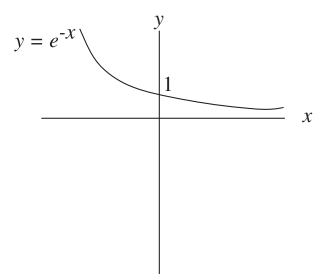
The y-intercept is $1 + 1 = 2$.

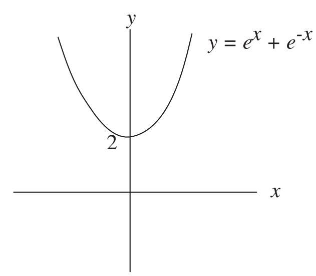
Interval	$x < 0$	$x = 0$	$x > 0$
$\frac{dy}{dx}$	< 0	$= 0$	> 0
Graph of $y = f(x)$	Decreasing	Local Min	Increasing

$\lim_{x \to -\infty} (e^x + e^{-x}) = \infty$

$\lim_{x \to \infty} (e^x + e^{-x}) = \infty$







d. $g(w) = \frac{4w^2 - 3}{w^3}$

$= \frac{4}{w} - \frac{3}{w^3},\ w \neq 0$

(i) $g'(w) = -\frac{4}{w^2} + \frac{9}{w^4}$

$= \frac{9 - 4w^2}{w^4}$

$g''(w) = \frac{8}{w^3} - \frac{36}{w^5}$

$= \frac{8w^2 - 36}{w^5}$

For possible points of inflection, we solve $g''(w) = 0$:

$8w^2 - 36 = 0$, since $w^5 \neq 0$

$w^2 = \frac{9}{2}$

$w = \pm \frac{3}{\sqrt{2}}.$

Interval	$w < -\frac{3}{\sqrt{2}}$	$w = -\frac{3}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}} < w < 0$	$< w < 0 < \frac{3}{\sqrt{2}}$	$w = \frac{3}{\sqrt{2}}$	$w > \frac{3}{\sqrt{2}}$
$g''(w)$	< 0	$= 0$	> 0	< 0	0	> 0
Graph of $g(w)$	Concave Down	Point of Inflection	Concave Up	Concave Down	Point of Inflection	Concave Up

The points of inflection are $\left(-\frac{3}{\sqrt{2}}, -\frac{8\sqrt{2}}{9}\right)$

and $\left(\frac{3}{\sqrt{2}}, \frac{8\sqrt{2}}{9}\right)$.

(ii) There is no y-intercept.

The x-intercept is $\pm \frac{3}{\sqrt{2}}$.

For critical values, we solve $g'(w) = 0$:

$9 - 4w^2 = 0$ since $w^4 \neq 0$

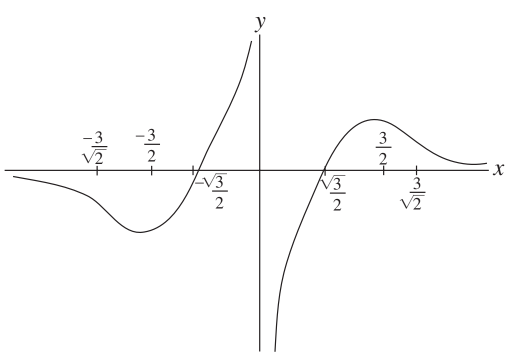
$$w = \pm \frac{3}{2}.$$

Interval	$w < -\frac{3}{2}$	$w = -\frac{3}{2}$	$-\frac{3}{2} < w < 0$	$0 < w < \frac{3}{2}$	$w = \frac{3}{2}$	$w > \frac{3}{2}$
$g'(w)$	< 0	$= 0$	> 0	> 0	$= 0$	< 0
Graph $g(wP)$	Decreasing	Local Min	Increasing	Increasing	Local Max	Decreasing

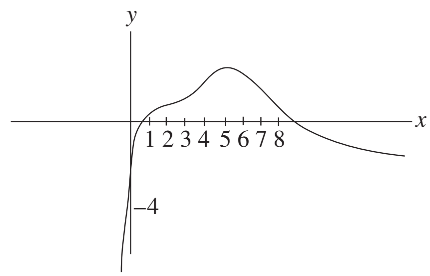
$$\lim_{w \to 0^-} \frac{4w^2 - 3}{w^3} = \infty, \lim_{w \to 0^+} \frac{4w^2 - 3}{w^3} = -\infty$$

$$\lim_{w \to -\infty} \left(\frac{4}{w} - \frac{3}{w^3}\right) = 0, \lim_{w \to \infty} \left(\frac{4}{w} - \frac{3}{w^3}\right) = 0$$

Thus, $y = 0$ is a horizontal asymptote and $x = 0$ is a vertical asymptote.



9. The graph is increasing when $x < 2$ and when $2 < x < 5$.
The graph is decreasing when $x > 5$.
The graph has a local maximum at $x = 5$.
The graph has a horizontal tangent line at $x = 2$.
The graph is concave down when $x < 2$ and when $4 < x < 7$.
The graph is concave up when $2 < x < 4$ and when $x > 7$.
The graph has points of inflection at $x = 2$, $x = 4$, and $x = 7$.
The y-intercept of the graph is -4.



10. $f(x) = ax^3 + bx^2 + c$

$f'(x) = 3ax^2 + 2bx$

$f''(x) = 6ax + 2b$

Since $(2, 11)$ is a relative extremum, $f(2) = 12a + 4b = 0$.

Since $(1, 5)$ is an inflection point, $f''(1) = 6a + 2b = 0$.

Since the points are on the graph,

$a + b + c = 5$ and

$\underline{8a + 4b + c = 11}$

$7a + 3b = 6$

$9a + 3b = 0$

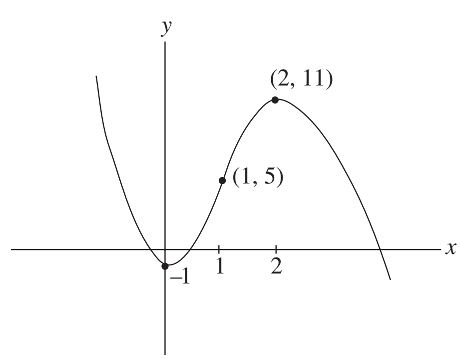
$2a = -6$

$a = -3$

$b = 9$

$c = -1$.

Thus, $f(x) = -3x^3 + 9x^2 - 1$.



11. $f(x) = (x+1)^{\frac{1}{2}} + bx^{-1}$

$f'(x) = \frac{1}{2}(x+1)^{-\frac{1}{2}} - bx^{-2}$

$f''(x) = -\frac{1}{4}(x+1)^{-\frac{3}{2}} + 2bx^{-3}$

Since the graph of $y = f(x)$ has a point of inflection at $x = 3$:

$$-\frac{1}{4}(4)^{-\frac{3}{2}} + \frac{2b}{27} = 0$$

$$-\frac{1}{32} + \frac{2b}{27} = 0$$

$$b = \frac{27}{64}.$$

12. $f(x) = ax^4 + bx^3$

$f'(x) = 4ax^3 + 3bx^2$

$f''(x) = 12ax^2 + 6bx$

For possible points of inflection, we solve $f''(x) = 0$:

$12ax^2 + 6bx = 0$

$6x(2ax + b) = 0$

$x = 0$ or $x = -\frac{b}{2a}$.

The graph of $y = f''(x)$ is a parabola with x-intercepts 0 and $-\frac{b}{2a}$.

We know the values of $f''(x)$ have opposite signs when passing through a root. Thus, at $x = 0$ and at $x = -\frac{b}{2a}$, the concavity changes as the graph goes through these points. Thus, $f(x)$ has points of inflection at $x = 0$ and $x = -\frac{b}{2a}$.

To find the x-intercepts, we solve $f(x) = 0$

$x^3(ax + b) = 0$

$x = 0$ or $x = -\frac{b}{a}$.

The point midway between the x-intercepts has x-coordinate $-\frac{b}{2a}$.

The points of inflection are $(0, 0)$ and $\left(-\frac{b}{2a}, -\frac{b^4}{16a^3}\right)$.

13. a. $y = \frac{x^3 - 2x^2 + 4x}{x^2 - 4} = x - 2 + \frac{8x - 8}{x^2 - 4}$ (by division of polynomials). The graph has discontinuities at $x = \pm 2$.

$$\left.\begin{aligned}\lim_{x\to -2^-}\left(x - 2 + \frac{8x-8}{x^2-4}\right) &= -\infty \\ \lim_{x\to -2^+}\left(x - 2 + \frac{8x-8}{x^2-4}\right) &= \infty\end{aligned}\right\} = -2 \text{ is a vertical asymptote.}$$

$$\left.\begin{aligned}\lim_{x\to 2^-}\left(x - 2 + \frac{8x-8}{x^2-4}\right) &= -\infty \\ \lim_{x\to 2^+}\left(x - 2 + \frac{8x-8}{x^2-4}\right) &= \infty\end{aligned}\right\} = 2 \text{ is a vertical asymptote.}$$

When $x = 0$, $y = 0$.

Also, $y = \frac{x(x^2 - 2x + 4)}{x^2 - 4} = \frac{x[(x-1)^2 + 3]}{x^2 - 4}$.

Since $(x - 1)^2 + 3 > 0$, the only x-intercept is $x = 0$.

Since $\lim_{x\to\infty} \frac{8x - 8}{x^2 - 4} = 0$, the curve approaches the value $x - 2$ as $x \to \infty$. This suggests that the line $y = x - 2$ is an oblique asymptote. It is verified by the limit $\lim_{x\to\infty} [x - 2 - f(x)] = 0$. Similarly, the curve approaches $y = x - 2$ as $x \to -\infty$.

$$\frac{dy}{dx} = 1 + \frac{8(x^2 - 4) - 8(x-1)(2x)}{(x^2-4)^2}$$

$$= 1 - \frac{8(x^2 - 2x + 4)}{(x^2-4)^2}$$

We solve $\frac{dy}{dx} = 0$ to find critical values:

$8x^2 - 16x + 32 = x^4 - 8x^2 + 16$

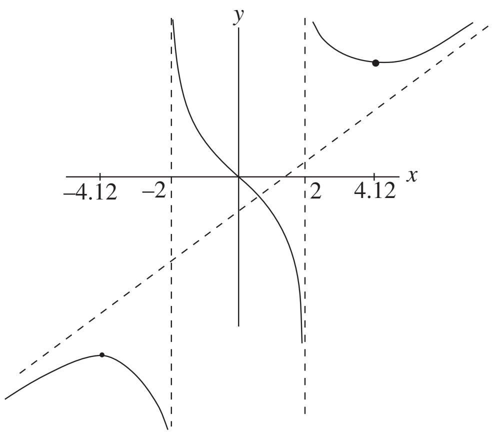
$x^4 - 16x^2 - 16 = 0$

$x^2 = 8 + 4\sqrt{5}$ ($8 - 4\sqrt{5}$ is inadmissible)

$x \doteq \pm 4.12$.

Interval	$x <$ -4.12	$x =$ -4.12	-4.12 $< x < 2$	$-2< x$ < 2	$2 < x$ < 4.12	$x =$ 4.12	$x >$ 4.12
$\frac{dy}{dx}$	> 0	$= 0$	< 0	< 0	< 0	0	> 0
Graph of y	Increasing	Local Max	Decreasing	Decreasing	Decreasing Min	Local	Increasing

$\lim\limits_{x \to \infty} y = \infty$ and $\lim\limits_{x \to -\infty} y = -\infty$



Exercise 9.5

1. a. $y = x^3 - 9x^2 + 15x + 30$

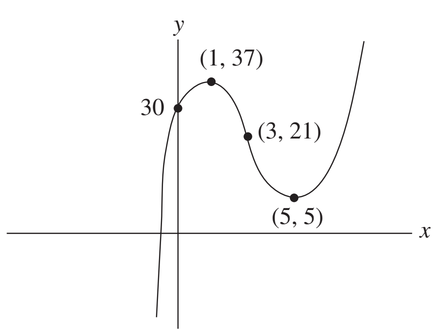
We know the general shape of a cubic polynomial with leading coefficient positive. The local extrema will help refine the graph.

$\frac{dy}{dx} = 3x^2 - 18x + 15$

Set $\frac{dy}{dx} = 0$ to find the critical values:

$$\begin{aligned} 3x^2 - 18x + 15 &= 0 \\ x^2 - 6x + 5 &= 0 \\ (x-1)(x-5) &= 0 \\ x = 1 \text{ or } x &= 5. \end{aligned}$$

The local extrema are (1, 37) and (5, 5).



b. $f(x) = 4x^3 + 18x^2 + 3$

The graph is that of a cubic polynomial with leading coefficient negative. The local extrema will help refine the graph.

$\frac{dy}{dx} = 12x^2 + 36x$

To find the critical values, we solve $\frac{dy}{dx} = 0$:

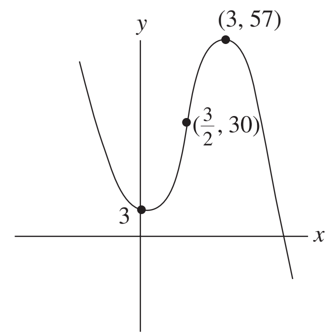
$-12x(x - 3) = 0$

$x = 0$ or $x = 3$.

The local extrema are (0, 3) and (3, 57).

$\frac{d^2y}{dx^2} = -24x + 36$

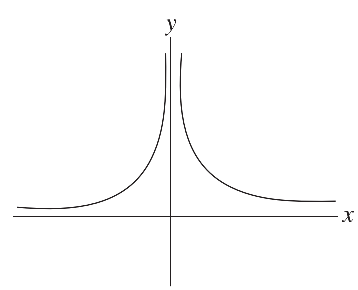
The point of inflection is $\left(\frac{3}{2}, 30\right)$.



c. $y = 3 + \frac{1}{(x+2)^2}$

We observe that $y = 3 + \frac{1}{(x+2)^2}$ is just a translation of $y = \frac{1}{x^2}$.

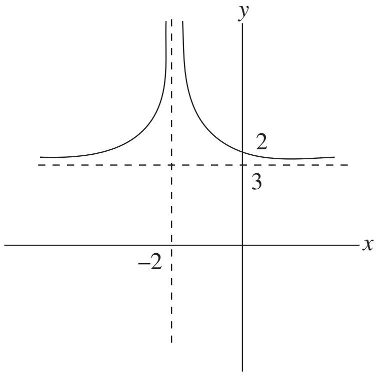
The graph of $y = \frac{1}{x^2}$ is



The reference point (0, 0) for $y = \frac{1}{x^2}$ becomes the point (–2, 3) for $y = 3 + \frac{1}{(x+2)^2}$. The vertical asymptote is $x = -2$, and the horizontal asymptote is $y = 3$.

$\frac{dy}{dx} = -\frac{2}{(x+2)^3}$, hence there are no critical points.

$\frac{d^2y}{dx^2} = \frac{6}{(x+2)^4} > 0$, hence the graph is always concave up.



d. $f(x) = x^4 - 4x^3 - 8x^2 + 48x$

We know the general shape of a fourth degree polynomial with leading coefficient positive. The local extrema will help refine the graph.

$f'(x) = 4x^3 - 12x^2 - 16x + 48$

For critical values, we solve $f'(x) = 0$

$$x^3 - 3x^2 - 4x + 12 = 0.$$

Since $f'(2) = 0$, $x - 2$ is a factor of $f'(x)$.

The equation factors are $(x-2)(x-3)(x+2) = 0$.

The critical values are $x = -2, 2, 3$.

$f''(x) = 12x^2 - 24x - 16$

Since $f''(-2) = 80 > 0$, $(-2, -80)$ is a local minimum.

Since $f''(2) = -16 < 0$, $(2, 48)$ is a local maximum.

Since $f''(3) = 20 > 0$, $(3, 45)$ is a local minimum.

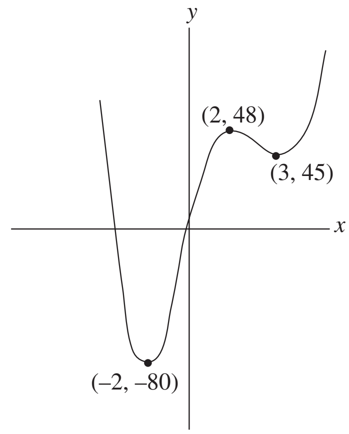
The graph has x-intercept*s* 0 and –3.2.

The points of inflection can be found by solving $f''(x) = 0$:

$3x^2 - 6x - 4 = 0$

$x = \frac{6 \pm \sqrt{84}}{6}$

$x \doteq -\frac{1}{2}$ or $\frac{5}{2}$.



e. $y = \frac{2x}{x^2 - 25}$

There are discontinuities at $x = -5$ and $x = 5$.

$\lim_{x \to -5^-} \left(\frac{2x}{x^2 - 25}\right) = -\infty$ and $\lim_{\to -5^+} \left(\frac{2x}{x^2 - 25}\right) = \infty$

$\lim_{x \to 5^-} \left(\frac{2x}{x^2 - 25}\right) = -\infty$ and $\lim_{\to 5^+} \left(\frac{2x}{x^2 - 25}\right) = \infty$

$x = -5$ and $x = 5$ are vertical asymptotes.

$\frac{dy}{dx} = \frac{2(x^2 - 25) - 2x(2x)}{(x^2 - 25)^2} = -\frac{2x^2 + 50}{(x^2 - 25)^2} < 0$ for

all x in the domain. The graph is decreasing throughout the domain.

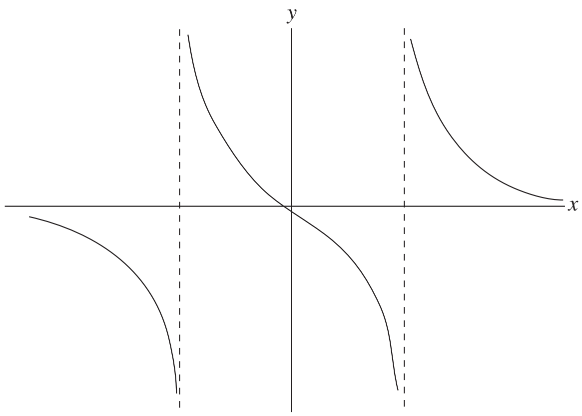
$$\left.\begin{aligned} \lim_{x \to \infty} \left(\frac{2x}{x^2 - 25}\right) &= \lim_{x \to \infty} \left(\frac{\frac{2}{x}}{1 - \frac{25}{x^2}}\right) \\ &= 0 \\ \lim_{x \to -\infty} \left(\frac{\frac{2}{x}}{1 - \frac{25}{x^2}}\right) &= 0 \end{aligned}\right\} y = 0 \text{ is a horizontal asymptote.}$$

$$\frac{d^2y}{dx^2} = -\frac{4x(x^2 - 25)^2 - (2x^2 + 50)(2)(x^2 - 25)(2x)}{(x^2 - 25)^4}$$

$$= \frac{4x^3 + 300x}{(x^2 - 25)^3} = \frac{4x(x^2 + 75)}{(x^2 - 25)^3}$$

There is a possible point of inflection at $x = 0$.

Interval	$x < -5$	$-5 < x < 0$	$x = 0$	$0 < x < 5$	$x > 5$
$\frac{d^2y}{dx^2}$	< 0	> 0	$= 0$	< 0	> 0
Graph of y	Concave Down	Concave Up	Point of Inflection	Concave Down	Concave Up



f. $y = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$

The graph of $y = f(x)$ is always above the x-axis. The y-intercept is $\frac{1}{\sqrt{2\pi}} \doteq 0.4$.

$$\frac{dy}{dx} = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} (-x)$$

$\frac{dy}{dx} = 0$ when $x = 0$. Thus, $(0, 0.4)$ is a critical point.

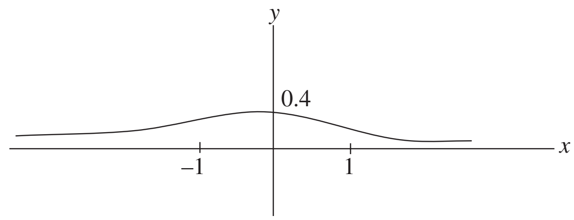
$$\frac{d^2y}{dx^2} = \frac{1}{\sqrt{2\pi}} \left(e^{-\frac{x^2}{2}} (-x)(-x) + e^{-\frac{x^2}{2}} (-1) \right)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} (x^2 - 1)$$

When $x = 0$, $\frac{d^2y}{dx^2} < 0$. Thus, $(0, 0.4)$ is a local maximum. Possible points of inflection occur when $x^2 - 1 = 0$ or $x = -1$ and $x = 1$.

Interval	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	> 1
$\frac{d^2y}{dx^2}$	> 0	$= 0$	< 0	$= 0$	> 0
Graph of y	Concave Up	Point of Inflection	Concave Down	Point of Inflection	Concave Up

$$\lim_{x \to \infty} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \right) = 0 \text{ and } \lim_{x \to \infty} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \right) = 0$$



g. $y = \frac{6x^2 - 2}{x^3}$

$= \frac{6}{x} - \frac{2}{x^3}$

There is a discontinuity at $x = 0$.

$$\lim_{x \to 0^-} \frac{6x^2 - 2}{x^3} = \infty \text{ and } \lim_{x \to 0^+} \frac{6x^2 - 2}{x^3} = -\infty$$

The y-axis is a vertical asymptote. There is no y-intercept. The x-intercept is $\pm \frac{1}{\sqrt{3}}$.

$$\frac{dy}{dx} = -\frac{6}{x^2} + \frac{6}{x^4} = \frac{-6x^2 + 6}{x^4}$$

$\frac{dy}{dx} = 0$ when $6x^2 = 6$

$x = \pm 1$

Interval	$x < -1$	$x = -1$	$-1 < x < 0$	$0 < x < 1$	$x = 1$	$x > 1$
$\frac{dy}{dx}$	< 0	$= 0$	> 0	> 0	$= 0$	< 0
Graph of $y = f(x)$	Decreasing	Local Min	Increasing	Increasing	Local Max	Decreasing

There is a local minimum at $(-1, -4)$ and a local maximum at $(1, 4)$.

$$\frac{d^2y}{dx^2} = \frac{12}{x^3} - \frac{24}{x^5} = \frac{12x^2 - 24}{x^5}$$

For possible points of inflection, we solve $\frac{d^2y}{dx^2} = 0$ $(x^5 \neq 0)$:

$12x^2 = 24$

$x = \pm \sqrt{2}$.

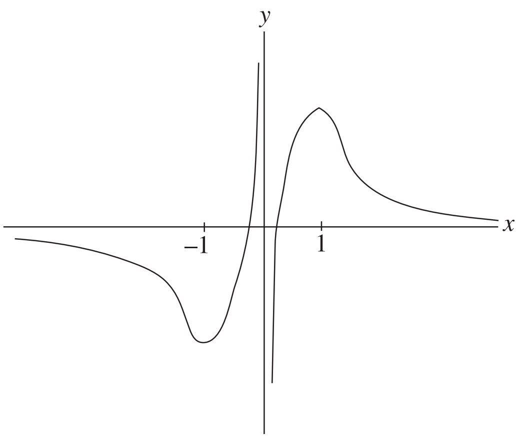
Interval	$x < -\sqrt{2}$	$x = -\sqrt{2}$	$-\sqrt{2} < x < 0$	$0 < x < \sqrt{2}$	$x = \sqrt{2}$	$x > \sqrt{2}$
$\frac{d^2y}{dx^2}$	< 0	$= 0$	> 0	< 0	$= 0$	> 0
Graph of $y = f(x)$	Concave Down	Point of Inflection	Concave Up	Concave Down	Point of Inflection	Concave Up

There are points of inflection at $\left(-\sqrt{2}, -\frac{5}{\sqrt{2}}\right)$ and $\left(\sqrt{2}, \frac{5}{\sqrt{2}}\right)$.

$$\lim_{x\to\infty} \frac{6x^2 - 2}{x^3} = \lim_{x\to\infty} \frac{\frac{6}{x} - \frac{2}{x^3}}{1} = 0$$

$$\lim_{x = -\infty} \frac{\frac{6}{x} - \frac{6}{x^3}}{x} = 0$$

The x-axis is a horizontal asymptote.



h. $s = \frac{50}{1 + 5e^{-0.01t}}, t \geq 0$

When $t = 0$, $s = \frac{50}{6}$.

$$\frac{ds}{dt} = 50(-1)(1 + 5e^{-0.01t})^{-2}\,(5e^{-0.01t})(-0.01)$$

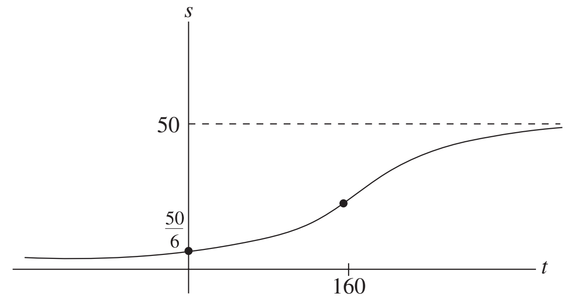
$$= \frac{2.5e^{-0.01t}}{(1 + 5e^{-0.01t})^2}$$

Since $\frac{ds}{dt} > 0$ for all t, s is always increasing.

$$\lim_{t\to\infty} \left(\frac{50}{1 + 5e^{-0.01t}}\right) = 50$$

$$\lim_{t\to-\infty} \left(\frac{50}{1 + 5e^{-0.01t}}\right) = 0$$

Thus, $s = 50$ is a horizontal asymptote for large values of t, and $s = 0$ is a horizontal asymptote for large negative values of t. It can be shown that there is a point of inflection at $t \doteq 160$.



i. $y = \frac{x + 3}{x^2 - 4}$

There are discontinuities at $x = -2$ and at $x = 2$.

$$\lim_{x\to-2^-} \left(\frac{x + 3}{x^2 - 4}\right) = \infty \text{ and } \lim_{x\to-2^+} \left(\frac{x + 3}{x^2 - 4}\right) = -\infty$$

$$\lim_{x\to 2^-} \left(\frac{x + 3}{x^2 - 4}\right) = -\infty \text{ and } \lim_{x\to 2^-} \left(\frac{x + 3}{x^2 - 4}\right) = \infty$$

There are vertical asymptotes at $x = -2$ and $x = 2$.

When $x = 0$, $y = -\frac{3}{4}$. The x-intercept is -3.

$$\frac{dy}{dx} = \frac{(1)(x^2 - 4) - (x + 3)(2x)}{(x^2 - 4)^2}$$

$$= \frac{-x^2 - 6x - 4}{(x^2 - 4)^2}$$

For critical values, we solve $\frac{dy}{dx} = 0$:

$x^2 + 6x + 4 = 0$

$$x = -\frac{6 \pm \sqrt{36 - 16}}{2}$$

$$= -3 \pm \sqrt{5}$$

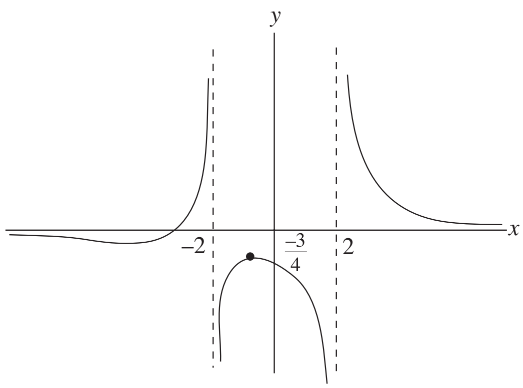
$$\doteq -5.2 \text{ or } -0.8.$$

Interval	$x < -5.2$	$x = -5.2$	$-5.2 < x < -2$	$-2 < x < -0.8$	$x = -0.8$	$-0.8 < x < 2$	$x > 2$
$\frac{dy}{dx}$	< 0	$= 0$	> 0	> 0	$= 0$	< 0	< 0
Graph of y	Decreasing	Local Min	Increasing	Increasing	Local Max	Decreasing	Decreasing

$$\lim_{x\to\infty} y = \lim_{x\to\infty}\left(\frac{\frac{1}{x}+\frac{3}{x^2}}{1-\frac{4}{x^2}}\right) = 0$$

$$\lim_{x\to-\infty}\left(\frac{\frac{1}{x}+\frac{3}{x^2}}{1-\frac{4}{x^2}}\right) = 0$$

The x-axis is a horizontal asymptote.



j. $y = \dfrac{x^2 - 3x + 6}{x - 1}$

$= x - 2 + \dfrac{4}{x-1}$

$$\begin{array}{r} x - 2 \\ x - 1 \overline{)\, x^2 - 3x + 6} \\ \underline{x^2 - x } \\ -2x + 6 \\ \underline{-2x + 2} \\ 4 \end{array}$$

There is a discontinuity at $x = 1$.

$$\lim_{x\to1^-}\left(\frac{x^2-3x+6}{x-1}\right) = -\infty$$

$$\lim_{x\to1^+}\left(\frac{x^2-3x+6}{x-1}\right) = \infty$$

Thus, $x = 1$ is a vertical asymptote.
The y-intercept is –6.
There are no x-intercepts ($x^2 - 3x + 6 > 0$ for all x in the domain).

$$\frac{dy}{dx} = 1 - \frac{4}{(x-1)^2}$$

For critical values, we solve $\dfrac{dy}{dx} = 0$:

$$1 - \frac{4}{(x-1)^2} = 0$$

$$\begin{aligned} (x-1)^2 &= 4 \\ x - 1 &= \pm 2 \\ x &= -1 \text{ or } x = 3. \end{aligned}$$

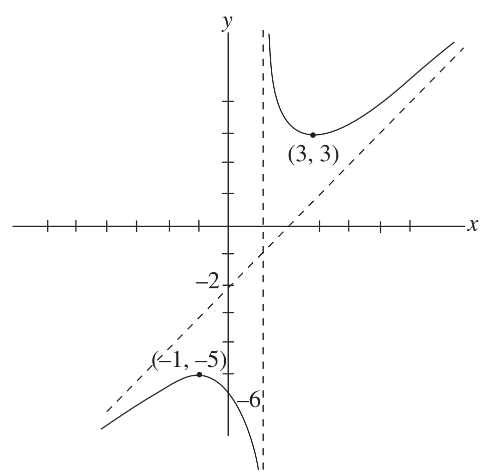
Interval	$x < -1$	$x = -1$	$-1 < x < 1$	$1 < x < 3$	$x = 3$	$x > 3$
$\frac{dy}{dx}$	> 0	$= 0$	< 0	< 0	$= 0$	> 0
Graph of y	Increasing	Local Max	Decreasing	Decreasing	Local Min	Increasing

$$\frac{d^2y}{dx^2} = \frac{8}{(x-1)^3}$$

For $x < 1$, $\dfrac{d^2y}{dx^2} < 0$ and y is always concave down.

For $x > 1$, $\dfrac{d^2y}{dx^2} > 0$ and y is always concave up.

The line $y = x - 2$ is an oblique asymptote.



k. $c = te^{-t} + 5$
When $t = 0$, $c = 5$.

$$\frac{dc}{dt} = e^{-t} - te^{-t} = e^{-t}(1 - t)$$

Since $e^{-t} - te^{-t} = e^{-t}(1 - t)$
Since $e^{-t} > 0$, the only value for which

$\dfrac{dc}{dt} = 0$ is $t = 1$.

Interval	$t < 1$	$t = 1$	$t > 1$
$\frac{dc}{dt}$	> 0	$= 0$	< 0
Graph of c	Increasing	Local Max	Decreasing

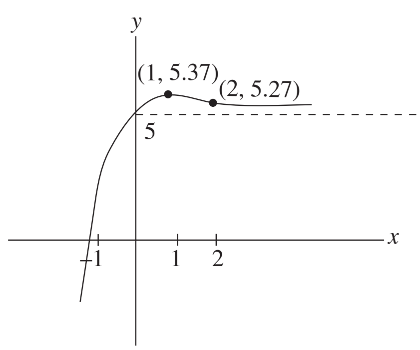
$\lim_{x \to \infty} (te^{-t} + t) = 5$

$\lim_{x \to -\infty} (te^{-t} + t) = -\infty$

$\frac{d^2c}{dt^2} = -e^{-t} - e^{-t} + te^{-t} = e^{-t}(t - 2)$

$\frac{d^2c}{dt^2} = 0$ when $t = 2$

Interval	$t < 2$	$t = 2$	$t > 0$
$\frac{d^2c}{dt^2}$	< 0	$= 0$	> 0
Graph of c	Concave Down	Point of Inflection	Concave Up



1. $y = x(\ln x)^3, x > 0$

$\frac{dy}{dx} = (\ln x)^3 + x(3)(\ln x)^2 \left(\frac{1}{x}\right) = (\ln x)^2(\ln x + 3)$

$\frac{dy}{dx} = 0$ when $\ln x = 0$ or $\ln x = -3$

$x = 1$ or $x = e^{-3} \doteq 0.05$

Interval	$0 < x < 0.05$	$x = 0.05$	$0.05 < x < 1$	$x = 1$	$x > 1$
$\frac{dy}{dx}$	< 0	$= 0$	> 0	0	> 0
Graph of y	Decreasing	Local Min	Increasing	Stationary Point	Increasing

There is no y-intercept. The x-intercept is 1.

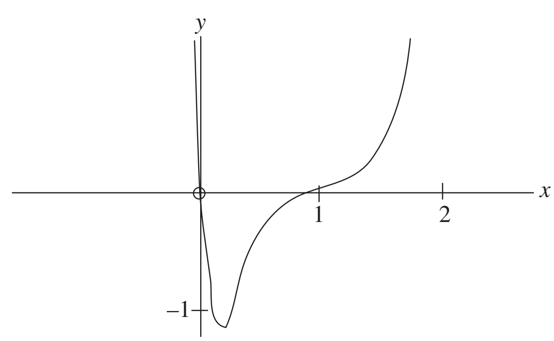
$\frac{d^2y}{dx^2} = 3(\ln x)^2 \left(\frac{1}{x}\right) + 6(\ln x)\left(\frac{1}{x}\right) = 3\,\frac{\ln x}{x}(\ln x + 2)$

$\frac{d^2y}{dx^2} = 0$ when $\ln x = 0$ or $\ln x = -2$

$x = 1$ or $x = e^{-2} \doteq 0.14$

Interval	$0 < x < 0.14$	$x = 0.14$	$0.14 < x < 1$	$x = 1$	$x > 1$
$\frac{d^2y}{dx^2}$	> 0	$= 0$	< 0	$= 0$	> 0
Graph of y	Concave Up	Point of Inflection	Concave Down	Point of Inflection	Concave Up

$\lim_{x \to \infty} [x(\ln x)^3] = \infty$



2. $y = ax^3 + bx^2 + cx + d$

Since (0, 0) is on the curve $d = 0$:

$\frac{dy}{dx} = 3ax^2 + 2bx + c$

At $x = 2$, $\frac{dy}{dx} = 0$.

Thus, $12a + 4b + c = 0$.

Since (2, 4) is on the curve, $8a + 4b + 2c = 4$ or $4a + 2b + c = 2$.

$$\frac{d^2y}{dx^2} = 6ax + 2b$$

Since (0, 0) is a point of inflection, $\frac{d^2y}{dx^2} = 0$ when $x = 0$.

Thus, $2b = 0$

$b = 0$.

Solving for a and c:

$$\begin{aligned} 12a + c &= 0 \\ 4a + c &= 2 \\ 8a &= -2 \\ a &= -\frac{1}{4} \\ c &= 3. \end{aligned}$$

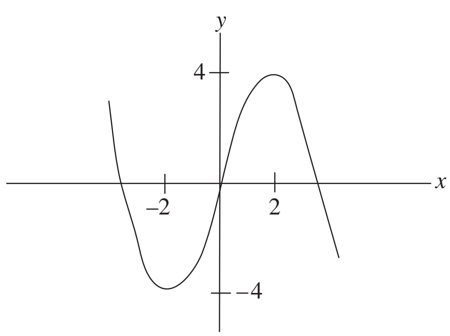
The cubic polynomial is $y = -\frac{1}{4}x^3 + 3x$.

The y-intercept is 0. The x-intercepts are found by setting $y = 0$:

$$-\frac{1}{4}x(x^2 - 12) = 0$$

$x = 0$, or $x = \pm 2\sqrt{3}$.

Let $y = f(x)$. Since $f(-x) = \frac{1}{4}x^3 - 3x = -f(x)$, $f(x)$ is an odd function. The graph of $y = f(x)$ is symmetric when reflected in the origin.



3. $g(x) = \frac{8e^x}{e^{2x}} + 4$

There are no discontinuities. The graph is always above the x-axis. The y-intercept is $\frac{8}{5}$.

$$g'(x) = \frac{8e^x(e^{2x} + 4) - 8e^x(e^{2x})2)}{(e^{2x} + 4)^2}$$

$$= \frac{8e^x(4 - e^{2x})}{(e^{2x} + 4)^2}$$

The only critical values occur when
$$\begin{aligned} 4 - e^{2x} &= 0 \\ e^{2x} &= 4 \\ 2x &= \ln 4 \\ x &= \ln \frac{4}{2} \\ &= \ln 2. \end{aligned}$$

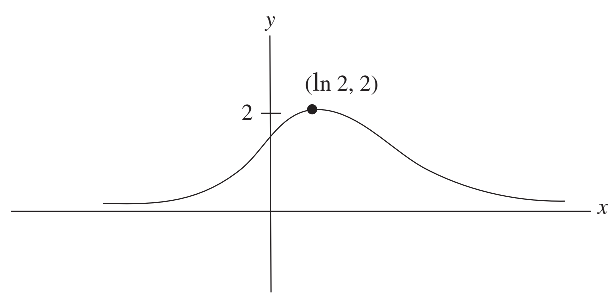
For $x < \ln 2$, $g'(x) > 0$

For $x > \ln 2$, $g'(x) < 0$

Thus, $(\ln 2, 2)$ is a local maximum point.

$$\left.\begin{aligned} &\lim_{x\to\infty}\left(\frac{8e^x}{e^{2x} + 4}\right) = \lim_{x\to\infty}\left(\frac{8}{e^x + \frac{4}{e^x}}\right) = 0 \\ &\lim_{x\to-\infty}\left(\frac{8e^x}{e^{2x} + 4}\right) = \frac{0}{0 + 4} = 0 \end{aligned}\right\}$$ Hence, the x-axis is a horizontal asymptote.

It is very cumbersome to evaluate $g''(x)$. Since there is a horizontal tangent line at the local maximum $(\ln 2, 2)$ and the x-axis is a horizontal asymptote, it is reasonable to conclude that there are two points of inflection. (It can be shown to be true.)



4. $y = e^x + \frac{1}{x}$

There is a discontinuity at $x = 0$.

$$\lim_{x\to0^-}\left(e^x + \frac{1}{x}\right) = -\infty \text{ and } \lim_{x\to0^+}\left(\frac{e^x + 1}{x}\right) = \infty$$

Thus, the y-axis is a vertical asymptote.

$$\frac{dy}{dx} = e^x - \frac{1}{x^2}$$

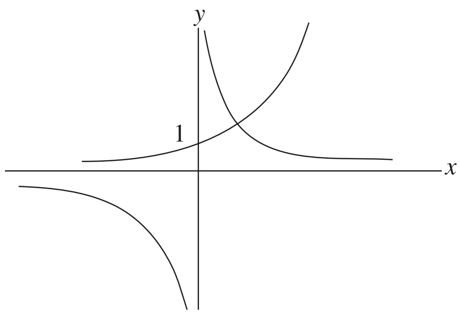
To find the critical values, we solve $\frac{dy}{dx} = 0$:

$$e^x - \frac{1}{x^2} = 0$$

This equation does not have a simple analytic solution. Solving $\frac{d^2y}{dx^2} = 0$ is even more cumbersome.

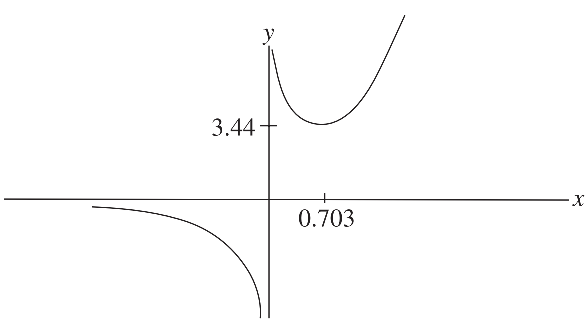
We use a different approach to sketch $y = e^x + \frac{1}{x}$.

We use the method of adding functions. The given function is the sum of $y_1 = e^x$ and $y_2 = \frac{1}{x}$.



For $x > 0$, the sum of the two functions is always positive. The resulting graph will be in the first quadrant. The graph of $y_2 = \frac{1}{x}$ dominates for values near 0, and the graph of $y_1 = e^x$ dominates for large values of x. It appears that this branch of the graph will have a relative minimum value. (A calculator solution of $\frac{dy}{dx} = 0$ verifies a relative minimum at $x \doteq 0.703$.)

For $x < 0$, the graph of $y_2 = \frac{1}{x}$ dominates the sum. There are no points of inflection.



5. $f(x) = \frac{k - x}{k^2 + x^2}$

There are no discontinuities.

The y-intercept is $\frac{1}{k}$ and the x-intercept is k.

$$f'(x) = \frac{(-1)(k^2 + x^2) - (k - x)(2x)}{(k^2 + x^2)^2}$$
$$= \frac{x^2 - 2kx - k^2}{(k^2 + x^2)^2}$$

For critical points, we solve $f'(x) = 0$:

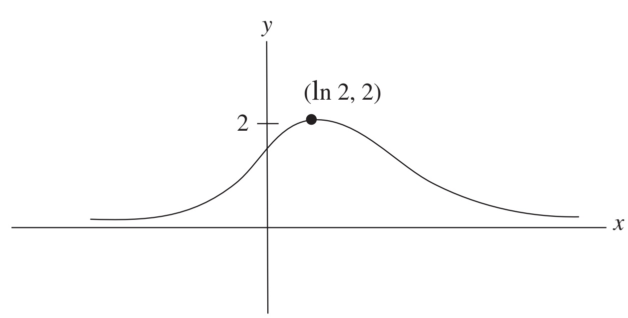
$$x^2 - 2kx - k^2 = 0$$
$$x^2 - 2kx + k^2 = 2k^2$$
$$(x - k)^2 = 2k^2$$
$$x - k = \pm\sqrt{2}\,k$$
$$x = (1 + \sqrt{2})k \text{ or } x = (1 - \sqrt{2})k.$$

Interval	$x < -0.41k$	$x \doteq 0.41k$	$-0.41k < x < 2.41k$	$x \doteq 2.41k$	$x > 2.41k$
$f'(x)$	> 0	$= 0$	< 0	$= 0$	> 0
Graph of $f(x)$	Increasing	Local Max	Decreasing	Local Min	Increasing

$$\lim_{x\to\infty}\left(\frac{k - x}{k^2 + x^2}\right) = \lim_{x\to\infty}\left(\frac{\frac{k}{x^2} - \frac{1}{x}}{\frac{k^2}{x^2} + 1}\right) = 0$$

$$\lim_{x\to-\infty}\left(\frac{\frac{k}{x^2} - \frac{1}{x}}{\frac{k^2}{x^2} + 1}\right) = 0$$

Hence, the x-axis is a horizontal asymptote.



6. $g(x) = x^{\frac{1}{3}}(x + 3)^{\frac{2}{3}}$

There are no discontinuities.

$$g'(x) = \frac{1}{3}x^{-\frac{2}{3}}(x + 3)^{\frac{2}{3}} + x^{\frac{1}{3}}\left(\frac{2}{3}\right)(x + 3)^{-\frac{1}{3}}(1)$$

$$= \frac{x + 3 + 2x}{3x^{\frac{2}{3}}(x + 3)^{\frac{1}{3}}} = \frac{3(x + 1)}{3x^{\frac{2}{3}}(x + 3)^{\frac{1}{3}}}$$

$$= \frac{x + 1}{x^{\frac{2}{3}}(x + 3)^{\frac{1}{3}}}$$

$g'(x) = 0$ when $x = -1$.

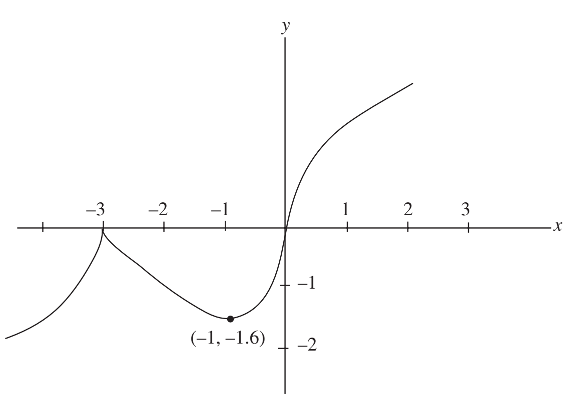
$g'(x)$ doesn't exist when $x = 0$ or $x = -3$.

Interval	$x < -3$	$x = -3$	$-3 < x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$	$x > 0$
$g'(x)$	> 0	Does Not Exist	< 0	$= 0$	> 0	Does Not Exist	> 0
Graph of $g(x)$	Increasing	Local Max	Decreasing	Local Min	Increasing		Increasing

There is a local maximum at (–3, 0) and a local minimum at (–1, –1.6). The second derivative is algebraically complicated to find. It can be verified that

$$g''(x) = \frac{-2}{x^{\frac{5}{3}}(x + 3)^{\frac{2}{3}}}.$$

Interval	$x < -3$	$x = -3$	$-3 < x < 0$	$x = 0$	$x > 0$
$g''(x)$	> 0	Does Not Exist	> 0	Does Not Exist	< 0
Graph $g(x)$	Concave Up	Cusp	Concave Up	Point of Inflection	Concave Down



7. a. $f(x) = \frac{x}{\sqrt{x^2 + 1}}$

$$= \frac{x}{|x|\sqrt{1 + \frac{1}{x^2}}}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{x\sqrt{1+\frac{1}{x^2}}}, \text{ since } x > 0$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1 + \frac{1}{x^2}}}$$

$$= 1$$

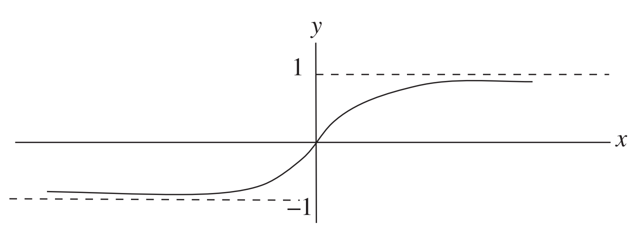
$y = 1$ is a horizontal asymptote to the right hand branch of the graph.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{-x\sqrt{1 + \frac{1}{x^2}}}, \text{ since } |x| = -x \text{ for } x < 0$$

$$= \lim_{x\to-\infty} \frac{1}{-\sqrt{1 + \frac{1}{x^2}}}$$

$$= -1$$

$y = -1$ is a horizontal asymptote to the left hand branch of the graph.



b. $g(t) = \sqrt{t^2 + 4t} - \sqrt{t^2 + t}$

$$= \frac{\left(\sqrt{t^2 + 4t} - \sqrt{t^2 + t}\right)\left(\sqrt{t^2 + 4t} + \sqrt{t^2 + t}\right)}{}$$

$$= \frac{3t}{\sqrt{t^2 + 4t} + \sqrt{t^2 + t}}$$

$$= \frac{3t}{|t|\sqrt{1 + \frac{4}{t}} + |t|\sqrt{1 + \frac{1}{t}}}$$

$$\lim_{t\to\infty} g(t) = \frac{3}{1+1} = \frac{3}{2}, \text{ since } |t| = t \text{ for } t > 0$$

$$\lim_{t\to-\infty} g(t) = \frac{3}{-1-1} = -\frac{3}{2}, \text{ since } |t| = -t \text{ for } t < 0$$

$y = \frac{3}{2}$ and $y = -\frac{3}{2}$ are horizontal asymptotes.

8. $y = ax^3 + bx^2 + cx + d$

$$\frac{dy}{dx} = 3ax^2 + 2bx + c$$

$$\frac{d^2y}{dx^2} = 6ax + 2b = 6a\left(x + \frac{b}{3a}\right)$$

For possible points of inflection, we solve $\frac{d^2y}{dx^2} = 0$:

$x = -\frac{b}{3a}$.

The sign of $\frac{d^2y}{dx^2}$ changes as x goes from values less than $\frac{-b}{3a}$ to values greater than $\frac{-b}{3a}$. Thus, there is a point of inflection at $x = \frac{-b}{3a}$.

At $x = \frac{b}{3a}$, $\frac{dy}{dx} = 3a\left(\frac{-b}{3a}\right)^2 + 2b\left(\frac{-b}{3a}\right) + c = c - \frac{b^2}{3a}$.

Review Exercise

1. a. $y = e^{nx}$

$$\frac{dy}{dx} = ne^{nx}$$

$$\frac{d^2y}{dx^2} = n^2e^{nx}$$

b. $f(x) = \ln(x+4)^{\frac{1}{2}}$

$$= \frac{1}{2}\ln(x+4)$$

$$f'(x) = \frac{1}{2} \bullet \frac{1}{x+4} = \frac{1}{2(x+4)}$$

$$f''(x) = -\frac{1}{2} \bullet \frac{1}{(x+4)^2} = -\frac{1}{2(x+4)^2}$$

c. $s = \frac{e^t - 1}{e^t + 1}$

$$\frac{ds}{dt} = \frac{e^t(e^t+1) - (e^t-1)(e^t)}{(e^t+1)^2}$$

$$= \frac{2e^t}{(e^t+1)^2}$$

$$\frac{d^2s}{dt^2} = \frac{2e^t(e^t+1)^2 - 2e^t(2)(e^t+1)(e^t)}{(e^t+1)^4}$$

$$= \frac{2e^{2t} + 2e^t - 4e^{2t}}{(e^t+1)^3}$$

$$= \frac{2e^t(1-e^t)}{(e^t+1)^3}$$

d. $g(t) = \ln\left(t + \sqrt{1+t^2}\right)$

$$g'(t) = \frac{1}{t + \sqrt{1+t^2}} \bullet \left(1 + \frac{1}{2}\left(1+t^2\right)^{-\frac{1}{2}}(2t)\right)$$

$$= \frac{1 + \frac{t}{\sqrt{1+t^2}}}{t + \sqrt{1+t^2}}$$

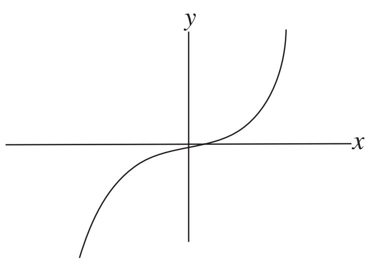
$$= \frac{\frac{\sqrt{1+t^2} + t}{\sqrt{1+t^2}}}{t + \sqrt{1+t^2}}$$

$$= \frac{1}{\sqrt{1+t^2}}$$

$$g''(t) = -\frac{1}{2}(1 + t^2)^{-\frac{3}{2}}(2t)$$

$$= \frac{-t}{(1 + t^2)^{\frac{3}{2}}}$$

3. No. A counter example is sufficient to justify the conclusion. The function $f(x) = x^3$ is always increasing yet the graph is concave down for $x < 0$ and concave up for $x > 0$.



4. a. $f(x) = -2x^3 + 9x^2 + 20$

$f'(x) = -6x^2 + 18x$

For critical values, we solve:

$f'(x) = 0$

$-6x(x - 3) = 0$

$x = 0$ or $x = 3$.

$f''(x) = -12x + 18$

Since $f''(0) = 18 > 0$, (0, 20) is a local minimum point. The tangent to the graph of $f(x)$ is horizontal at (0, 20). Since $f''(3) = -18 < 0$, (3, 47) is a local maximum point. The tangent to the graph of $f(x)$ is horizontal at (3, 47).

b. $g(t)\ \frac{e^{-2t}}{t^2}$

$g(t) = e^{-2t}t^{-2},\ t \neq 0$

$g'(t) = -2e^{-2t}t^{-2} + e^{-2t}(-2t^{-3})$

$= -\frac{2e^{-2t}(t + 1)}{t^3}$

Since $e^{-2t} > 0$ for all t, and $g(t)$ has a discontinuity at $t = 0$, the only critical value is $t = -1$.

Interval	$t < -1$	$t = -1$	$-1 < t < 0$	$t > 0$
$g'(t)$	< 0	$= 0$	> 0	< 0
Graph of $g(t)$	Decreasing	Local Min	Increas-ing	Decreas-ing

There is a local minimum at $(-1, e^2)$. The tangent line at $(-1, e^2)$ is parallel to the x-axis.

c. $h(x) = \frac{x - 3}{x^2 + 7}$

$h'(x) = \frac{(1)(x^2 + 7) - (x - 3)(2x)}{(x^2 + 7)^2}$

$= \frac{7 + 6x - x^2}{(x^2 + 7)^2}$

$= \frac{(7 - x)(1 + x)}{(x^2 + 7)^2}$

Since $x^2 + 7 > 0$ for all x, the only critical values occur when $h'(x) = 0$. The critical values are $x = 7$ and $x = -1$.

Interval	$x < -1$	$x = -1$	$-1 < x < 7$	$x = 7$	$x > 7$
$h'(x)$	< 0	$= 0$	> 0	$= 0$	< 0
Graph of $h(t)$	Decreas-ing	Local Min	Increas-ing	Local Max	Decreas-ing

There is a local minimum at $\left(-1, -\frac{1}{2}\right)$ and a local maximum at $\left(7, \frac{1}{14}\right)$. At both points, the tangents are parallel to the x-axis.

d. $k(x) = \ln(x^3 - 3x^2 - 9x)$

The domain of $k(x)$ is the set of all x such that $x^3 - 3x^2 - 9x > 0$.

Let $g(x) = x^2 - 3x^2 - 9x$.

The x-intercepts of the graph of $g(x)$ are found by solving $g(x) = 0$:

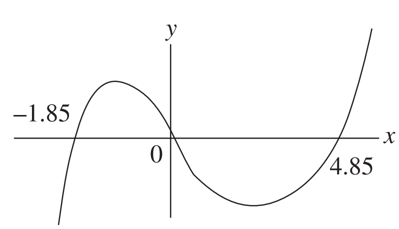
$x(x^2 - 3x - 9) = 0$

$x = 0$ or $x = \frac{3 \pm \sqrt{9 + 36}}{2}$

$= \frac{3 \pm 3\sqrt{5}}{2}$

$= 4.85$ or -1.85.

The graph of $y = g(x)$ is



Thus, the domain of $k(x)$ is $-1.85 < x < 0$ or $x > 4.85$.

$k'(x) = \dfrac{3x^2 - 6x - 9}{x^3 - 3x^2 - 9x}$.

Since the denominator $x^3 - 3x^2 - 9x > 0$, the only critical values of $k(x)$ result from

$$3x^2 - 6x - 9 = 0$$
$$x^2 - 2x - 3 = 0$$
$$(x - 3)(x + 1) = 0$$

$x = -1$ or $x = 3$ (this value is not in the domain).

Interval	$-1.85 < x < -1$	$x = -1$	$-1 < x < 0$	$x > 4.85$
$k'(x)$	> 0	$= 0$	< 0	> 0
Graph of $k(x)$	Increasing	Local Max	Decreasing	Increasing

Thus, $(-1, \ln 5)$ is a local maximum. The tangent line is parallel to the x-axis at $(-1, \ln 5)$.

6. **a.** $y = \dfrac{2x}{x - 3}$

There is a discontinuity at $x = 3$.

$\lim\limits_{x\to 3^-}\left(\dfrac{2x}{x-3}\right) = -\infty$ and $\lim\limits_{x\to 3^+}\left(\dfrac{2x}{x-3}\right) = \infty$

Therefore, $x = 3$ is a vertical asymptote.

b. $g(x) = \dfrac{x - 5}{x + 5}$

There is a discontinuity at $x = -5$.

$\lim\limits_{x\to -5^-}\left(\dfrac{x-5}{x+5}\right) = \infty$ and $\lim\limits_{x\to -5^+}\left(\dfrac{x-5}{x+5}\right) = -\infty$

Therefore, $x = -5$ is a vertical asymptote.

c. $s = \dfrac{s}{2e^x - 8}$

There is a discontinuity when $2e^x - 8 = 0$ or $x = \ln 4$.

$\lim\limits_{x\to \ln 4^-}\left(\dfrac{5}{2e^x-8}\right) = -\infty$ and $\lim\limits_{x\to \ln 4^+}\left(\dfrac{5}{2e^x-8}\right) = \infty$

Therefore, $x = \ln 4$ is a vertical asymptote.

d. $f(x) = \dfrac{x^2 - 2x - 15}{x + 3}$

$= \dfrac{(x+3)(x-5)}{x+3}$

$= x - 5, x \neq -3$

There is a discontinuity at $x = -3$.

$\lim\limits_{x\to -3^+} f(x) = -8$ and $\lim\limits_{x\to -3^-} f(x) = -8$

There is a hole in the graph of $y = f(x)$ at $(-3, -8)$.

7. **a.** $f(w) = \dfrac{\ln w^2}{w}$

$= (2\ln|w|)(w^{-1})$

$f'(w) = \left(\dfrac{2}{w}\right)(w^{-1}) + (2\ln|w|)(-w^{-2})$

$= 2w^{-2} - 2w^{-2}\ln|w|$

$f''(w) = -4w^{-3} + 4w^{-3}\ln|w| - 2w^{-2}\left(\dfrac{1}{w}\right)$

$= -6w^{-3} + 4w^{-3}\ln|w|$

$= \dfrac{4\ln|w| - 6}{w^3}$

For possible points of inflection, we solve $f''(w) = 0$.

Note: $w^3 \neq 0$.

$4\ln|w| = 6$

$w = \pm e^{\frac{3}{2}}$

Interval	$w < -e^{\frac{3}{2}}$	$w = -e^{\frac{3}{2}}$	$-e^{\frac{3}{2}} < w < 0$	$0 < w < e^{\frac{3}{2}}$	$w = e^{\frac{3}{2}}$	$w > e^{\frac{3}{2}}$
$f''(w)$	< 0	$= 0$	> 0	< 0	$= 0$	> 0
Graph $f(w)$	Concave Down	Point of Inflection	Concave Up	Concave Down	Point of Inflection	Concave Up

The points of inflection are $\left(-e^{\frac{3}{2}}, -\dfrac{3}{e^{\frac{3}{2}}}\right)$ and $\left(e^{\frac{3}{2}}, \dfrac{3}{e^{\frac{3}{2}}}\right)$.

b. $g(t) = te^t$

$g'(t) = e^t + te^t$

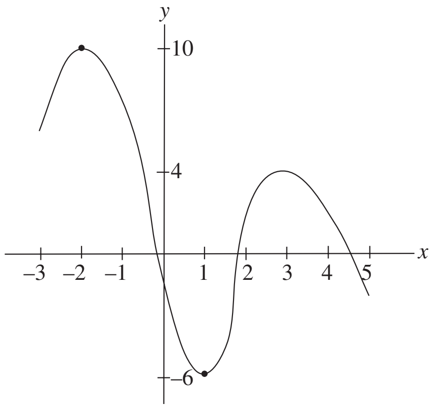
$g''(t) = e^t + e^t + te^t = e^t(t + 2)$

Since $e^t > 0$, $g''(t) = 0$ when $t = -2$.

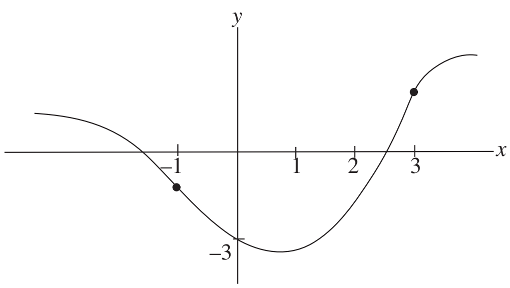
Interval	$t < -2$	$t = -2$	$t > -2$
$g''(t)$	< 0	$= 0$	> 0
Graph of $g(t)$	Concave Down	Point of Inflection	Concave Up

There is a point of inflection at $\left(-2, -\dfrac{2}{e^2}\right)$

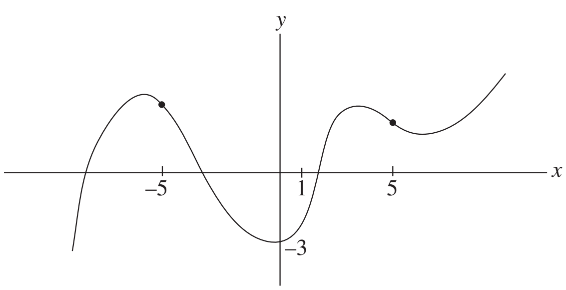
8.



9. c. (i)



(ii)



10. a. $g(x) = \dfrac{ax + b}{(x - 1)(x - 4)}$

$$= \frac{ax + b}{x^2 - 5x + 4}$$

$$g'(x) = \frac{a(x^2 - 5x + 4) - (ax + b)(2x - 5)}{(x^2 - 5x + 4)^2}$$

Since the tangent at (2, –1) has slope 0, $g'(2) = 0$.

Hence, $\dfrac{-2a + 2a + b}{4} = 0$ and $b = 0$.

Since (2, –1) is on the graph of $g(x)$:

$$-1 = \frac{2a + b}{-2}$$

$2a + 0 = 2$

$a = 1.$

Therefore $g(x) = \dfrac{x}{(x - 1)(x - 4)}$.

b. There are discontinuities at $x = 1$ and at $x = 4$.

$\lim\limits_{x \to 1^-} g(x) = \infty$ and $\lim\limits_{x \to 1^+} g(x) = -\infty$

$\lim\limits_{x \to 4^-} g(x) = -\infty$ and $\lim\limits_{x \to 4^+} g(x) = \infty$

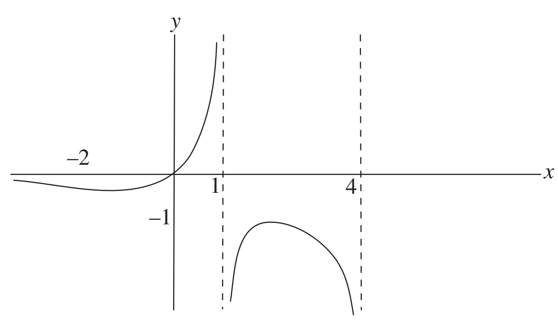
$x = 1$ and $x = 4$ are vertical asymptotes.
The y-intercept is 0.

$$g'(x) = \frac{4 - x^2}{(x^2 - 5x + 4)^2}$$

$g'(x) = 0$ when $x = \pm 2$.

Interval	$x < -2$	$x = -2$	$-2 < x < 1$	$1 < x < 2$	$x = 2$	$2 < x < 4$	$x > 4$
$g'(x)$	< 0	0	> 0	> 0	0	< 0	< 0
Graph of $g(x)$	Decreasing	Local Min	Increasing	Increasing	Local Max	Decreasing	Decreasing

There is a local minimum at $\left(-2, -\dfrac{1}{9}\right)$ and a local maximum at (2, –1).



11. a. $f(x) = \dfrac{2x^2 - 7x + 5}{2x - 1}$

$f(x) = x - 3 + \dfrac{2}{2x - 1}$

The equation of the oblique asymptote is $y = x - 3$.

$$\begin{array}{r|l} & x - 3 \\ 2x - 1 & 2x^2 - 7x + 5 \\ & \underline{2x^2 - x} \\ & \quad -6x + 5 \\ & \quad \underline{-6x + 3} \\ & \qquad\quad 2 \end{array}$$

$$\lim_{x\to\infty} [y - f(x)] = \lim_{x\to\infty} \left[x - 3 - \left(x - 3 + \frac{2}{2x - 1}\right)\right]$$

$$= \lim_{x\to\infty} \left[-\frac{2}{2x - 1}\right] = 0$$

b. $f(x) = \dfrac{4x^3 - x^2 - 15x - 50}{x^2 - 3x}$

$f(x) = 4x + 11 + \dfrac{18x - 50}{x^2 - 3x}$

$$\begin{array}{r|l} & 4x + 11 \\ x^2 - 3x & 4x^3 - x^2 - 15x - 50 \\ & \underline{4x^3 - 12x^2} \\ & \qquad 11x^2 - 15x \\ & \qquad \underline{11x^2 - 33x} \\ & \qquad\qquad 18x - 50 \end{array}$$

$\lim_{x\to\infty} [y - f(x)]$

$$= \lim_{x\to\infty} \left[4x + 11 - \left(4x + 11 + \frac{18x - 50}{x^2 - 3x}\right)\right]$$

$$= \lim_{x\to\infty} \left[\frac{\frac{18}{x} - \frac{50}{x^2}}{1 - \frac{3}{x}}\right]$$

$= 0$

12. a. $y = x^4 - 8x^2 + 7$

This is a fourth degree polynomial and is continuous for all x. The y-intercept is 7.

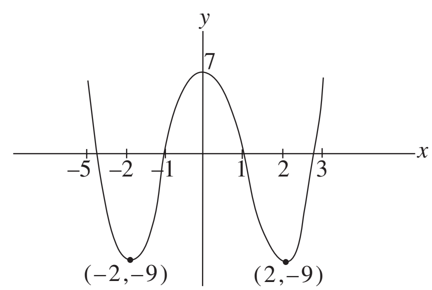
$$\frac{dy}{dx} = 4x^3 - 16x$$

$$= 4x(x - 2)(x + 2)$$

The critical values are $x = 0$, -2, and 2.

Interval	$x < -2$	$x = -2$	$-2 < x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$x > 2$
$\frac{dy}{dx}$	< 0	$= 0$	> 0	$= 0$	< 0	$= 0$	> 0
Graph of y	Decreasing	Local Min	Increasing	Local Max	Decreasing	Local Min	Increasing

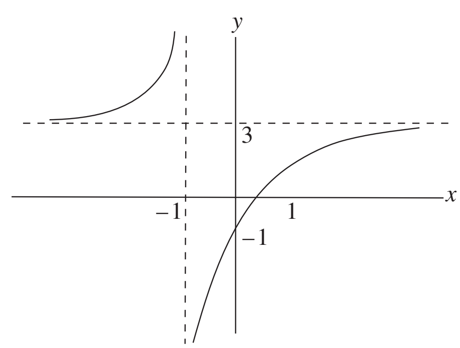
There are local minima at $(-2, -9)$ and at $(2, -9)$, and a local maximum at $(0, 7)$.



b. $f(x) = \dfrac{3x - 1}{x + 1}$

$= 3 - \dfrac{4}{x + 1}$

From experience, we know the graph of $y = -\dfrac{1}{x}$ is



The graph of the given function is just a transformation of the graph of $y = -\dfrac{1}{x}$. The vertical asymptote is $x = -1$ and the horizontal asymptote is $y = 3$. The y-intercept is -1 and there is an x-intercept at $\dfrac{1}{3}$.

c. $g(x) = \dfrac{x^2 + 1}{4x^2 - 9}$

$$= \frac{x^2 + 1}{(2x - 3)(2x + 3)}$$

The function is discontinuous at $x = -\dfrac{3}{2}$ and at $x = \dfrac{3}{2}$.

$$\lim_{x \to -\frac{3}{2}^-} g(x) = \infty$$

$$\lim_{x \to -\frac{3}{2}^+} g(x) = -\infty$$

$$\lim_{x \to \frac{3}{2}^-} g(x) = -\infty$$

$$\lim_{x \to -\frac{3}{2}^+} g(x) = \infty$$

Hence, $x = -\dfrac{3}{2}$ and $x = \dfrac{3}{2}$ are vertical asymptotes.

The y-intercept is $-\dfrac{1}{9}$.

$$g'(x) = \frac{2x(4x^2 - 9) - (x^2 + 1)(8x)}{(4x^2 - 9)^2} = \frac{-26x}{(4x^2 - 9)^2}$$

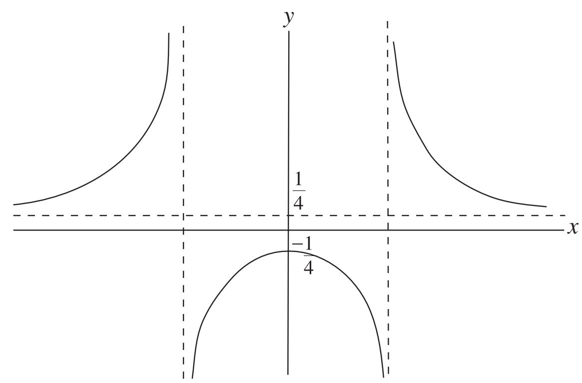
$g'(x) = 0$ when $x = 0$.

Interval	$x < -\frac{3}{2}$	$-\frac{3}{2} < x < 0$	$x = 0$	$0 < x < \frac{3}{2}$	$x > \frac{3}{2}$
$g'(x)$	> 0	> 0	$= 0$	< 0	< 0
Graph $g(x)$	Increasing	Increasing	Local Max	Decreasing	Decreasing

There is a local maximum at $\left(0, -\dfrac{1}{9}\right)$.

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \frac{1 + \frac{1}{x^2}}{4 - \frac{9}{x^2}} = \frac{1}{4} \text{ and } \lim_{x \to -\infty} g(x) = \frac{1}{4}$$

Hence, $y = \dfrac{1}{4}$ is a horizontal asymptote.



d. $y = 3x^2 \ln x,\ x > 0$

$$\frac{dy}{dx} = 6x \ln x + 3x^2\left(\frac{1}{x}\right) = 3x(2 \ln x + 1)$$

Since $x > 0$, the only critical value is when $2 \ln x + 1 = 0$

$$\ln x = -\frac{1}{2}$$

$$x = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}.$$

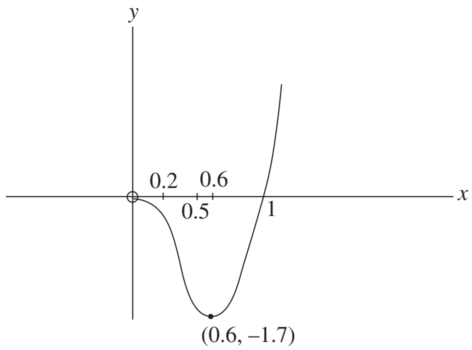
Interval	$0 < x < \frac{1}{\sqrt{e}}$	$x = \frac{1}{\sqrt{e}}$	$x > \frac{1}{\sqrt{e}}$
$\frac{dy}{dx}$	< 0	$= 0$	> 0
Graph of y	Decreasing	Local Min	Increasing

$$\frac{d^2y}{dx^2} = 6 \ln x + 6x\left(\frac{1}{x}\right) + 3 = 6 \ln x + 9$$

$$\frac{d^2y}{dx^2} = 0 \text{ when } \ln x = -\frac{3}{2}$$

$$x = e^{-\frac{3}{2}}$$

Interval	$0 < x < e^{-\frac{3}{2}}$	$x = e^{-\frac{3}{2}}$	$x > e^{-\frac{3}{2}}$
$\frac{d^2y}{dx^2}$	< 0	$= 0$	> 0
Graph of y	Concave Down	Point of Inflection	Concave Up



e. $h(x) = \dfrac{x}{x^2 - 4x + 4}$

$= \dfrac{x}{(x-2)^2} = x(x-2)^{-2}$

There is a discontinuity at $x = 2$

$\lim\limits_{x \to 2^-} h(x) = \infty = \lim\limits_{x \to 2^+} h(x)$

Thus, $x = 2$ is a vertical asymptote. The y-intercept is 0.

$h'(x) = (x-2)^{-2} + x(-2)(x-2)^{-3}(1)$

$= \dfrac{x - 2 - 2x}{(x-2)^3}$

$= \dfrac{-2 - x}{(x-2)^3}$

$h'(x) = 0$ when $x = -2$.

Interval	$x < -2$	$x = -2$	$-2 < x < 2$	$x > 2$
$h'(x)$	< 0	$= 0$	> 0	< 0
Graph of $h(x)$	Decreasing	Local Min	Increasing	Decreasing

There is a local minimum at $(-2, -\frac{1}{8})$.

$$\lim_{x \to \infty} h(x) = \lim_{x \to \infty} \frac{\frac{1}{x}}{1 - \frac{4}{x} + \frac{4}{x^2}} = 0$$

Similarly, $\lim\limits_{x \to -\infty} h(x) = 0$

The x-axis is a horizontal asymptote.

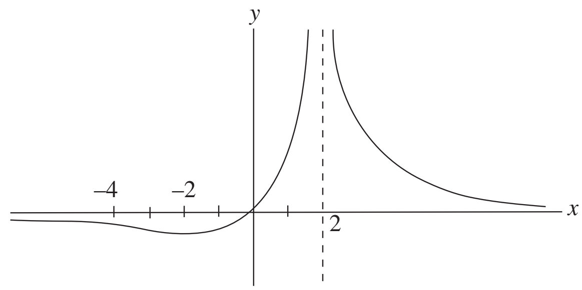
$h''(x) = -2(x-2)^{-3} - 2(x-2)^{-3} + 6x(x-2)^{-4}$

$= -4(x-2)^{-3} + 6x(x-2)^{-4}$

$= \dfrac{2x + 8}{(x-2)^4}$

$h''(x) = 0$ when $x = -4$

The second derivative changes signs on opposite sides of $x = -4$. Hence, $\left(-4, -\frac{1}{9}\right)$ is a point of inflection.



f. $f(t) = \dfrac{t^2 - 3t + 2}{t - 3}$

$= t + \dfrac{2}{t - 3}$

Thus, $f(t) = t$ is an oblique asymptote. There is a discontinuity at $t = 3$.

$\lim\limits_{t \to 3^-} f(t) = -\infty$ and $\lim\limits_{t \to 3^+} f(t) = \infty$

Therefore, $x = 3$ is a vertical asymptote.

The y-intercept is $-\frac{2}{3}$.

The x-intercepts are $t = 1$ and $t = 2$.

$f'(t) = 1 - \dfrac{2}{(t-3)^2}$

$f'(t) = 0$ when $1 - \dfrac{2}{(t-3)^2} = 0$

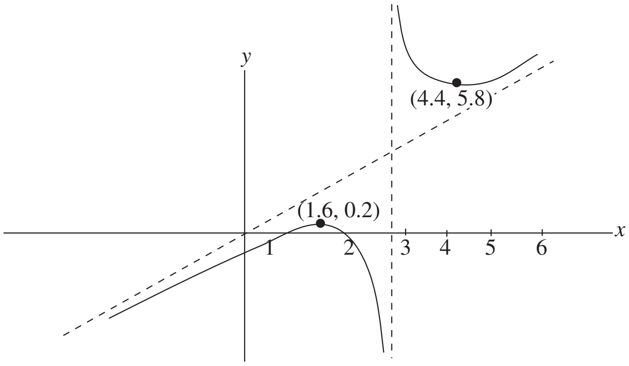
$(t-3)^2 = 2$

$t - 3 = \pm\sqrt{2}$

$t = 3 \pm \sqrt{2}$.

Interval	$t < 3 - \sqrt{2}$	$t = 3 - \sqrt{2}$	$3 - \sqrt{2} < t < 3$	$3 < t < 3 + \sqrt{2}$	$t = 3 + \sqrt{2}$	$t > 3 + \sqrt{2}$
$f'(t)$	> 0	$= 0$	< 0	< 0	$= 0$	> 0
Graph of $f(t)$	Increasing	Local Max	Decreasing	Decreasing	Local Min	Increasing

(1.6, 0.2) is a local maximum and (4.4, 5.8) is a local minimum.



g. $s = te^{-3t} + 10$

At $t = 0$, $s = 10$.

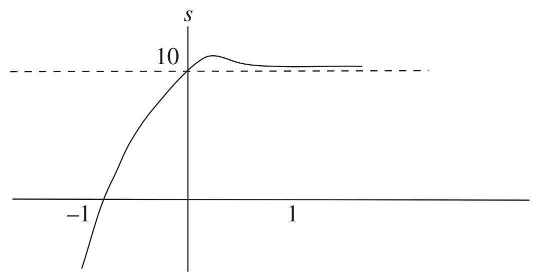
$$\frac{ds}{dt} = e^{-3t} + te^{-3t}(-3) = e^{-3t}(1 - 3t)$$

Since $e^{-3t} > 0$, $\frac{ds}{dt} = 0$ when $t = \frac{1}{3}$.

Interval	$t < \frac{1}{3}$	$t = \frac{1}{3}$	$t > \frac{1}{3}$
$\frac{ds}{dt}$	> 0	0	< 0
Graph of s	Increasing	Total Maximum	Decreasing

$\left(\frac{1}{3}, 10 + \frac{1}{3e}\right)$ is a local maximum point.

Since s is always decreasing for $t > \frac{1}{3}$, and te^{-3t} is positive for $t > \frac{1}{3}$, the graph will always be above the line $s = 10$, but it is approaching the line $s = 10$ as $t \to \infty$. Thus, $s = 10$, it is a horizontal asymptote. Since s is continuous for all t, has a local maximum at $\left(\frac{1}{3}, 10 + \frac{1}{3}e\right)$, and has $s = 10$ as a horizontal asymptote, we conclude that there is an inflection point at a value of $t > \frac{1}{3}$. (It can be shown that there is an inflection point at $t = \frac{2}{3}$.)



h. $P = \dfrac{100}{1 + 50e^{-0.2t}}$

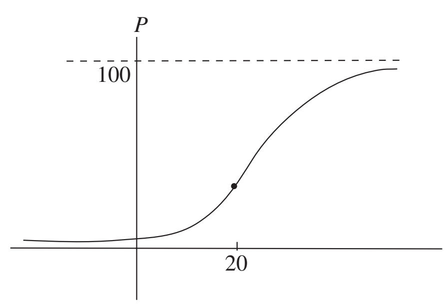
When $t = 0$, $P = \frac{100}{51} \doteq 1.99$

$$\frac{dP}{dt} = -100(1 + 50e^{-0.2t})^{-2}(50e^{-0.2t})(-0.2)$$
$$= \frac{1000e^{-0.2t}}{(1 + 50e^{-0.2t})^2}$$

Since $\frac{dP}{dt} > 0$ for all t, the graph is always increasing.

$$\lim_{t\to\infty}\left(\frac{100}{1 + 50e^{-0.2t}}\right) = 100 \text{ and } \lim_{t\to-\infty}\left(\frac{100}{1 + 50e^{-0.2t}}\right) = 0$$

Thus, $P = 100$ is a horizontal asymptote for large positive values of t, and $P = 0$ (the horizontal axis) is a horizontal asymptote for large negative values of t. It can be shown that there is a point of inflection at $t \doteq 20$.



13. $P = 10^4te^{-0.2t} + 100,\ t \geq 0$

a. $\dfrac{dP}{dt} = 10^4\left[e^{-0.2t} + te^{-0.2t}(-0.2)\right]$

$= 10^4e^{-0.2t}\left[1 - 0.2t\right]$

$\dfrac{dP}{dt} = 0$ when $t = \dfrac{1}{0.2} = 5.$

Since $\dfrac{dP}{dt} > 0$ for $0 \leq t < 5$ and $\dfrac{dP}{dt} < 0$ for $t > 5$,

the maximum population of the colony is $P = 10^4(5)e^{-1} \doteq 18\,994$ and it occurs on the fifth day after the creation of the colony.

b. The growth rate of the colony is the function $\dfrac{dP}{dt}$. The rate of change of the growth rate is

$\dfrac{d^2P}{dt^2} = 10^4\left[e^{-0.2t}(-0.2)(1 - 0.2t) + e^{-0.2t}(-0.2)\right]$

$= 10^4e^{-0.2t}[0.04t - 0.4].$

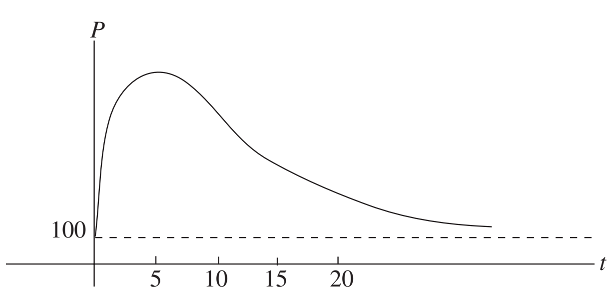
To determine when $\dfrac{d^2P}{dt^2}$ starts to increase, we need $\dfrac{d^3P}{dt^3}$.

$\dfrac{d^3P}{dt^3} = 10^4\left[e^{-0.2t}(-0.2)(0.04t - 0.4) + e^{-0.2t}(0.04)\right]$

$= 10^4e^{-0.2t}\left[0.12 - 0.008t\right]$

$= 80e^{-0.2t}\,(15 - t)$

Since $\dfrac{d^3P}{dt^3} > 0$ for $0 \leq t < 15$ and $\dfrac{d^3P}{dt^3} < 0$ for $t > 15$, $\dfrac{d^2P}{dt^2}$ is increasing from the moment the colony is formed and continues for the first 15 days.



14. $y = \ln\left[\dfrac{x^2+1}{x^2-1}\right], \dfrac{x^2+1}{x^2-1} > 0$

Since $x^2 + 1 > 0$ for all x, for y to be defined, $x^2 - 1 > 0$. The domain is $x < -1$ or $x > 1$.
y can be written as $y = \ln(x^2 + 1) - \ln(x^2 - 1)$.

Thus, $\dfrac{dy}{dx} = \dfrac{2x}{x^2+1} - \dfrac{2x}{x^2-1}$

$= \dfrac{-4x}{x^4-1} = -4x(x^4 - 1)^{-1}$

$\dfrac{d^2y}{dx^2} = -4(x^4 - 1)^{-1} - 4x(-1)(x^4 - 1)^{-2}(4x^3)$

$= \dfrac{-4x^4 + 4 + 16x^4}{(x^4-1)^2} = \dfrac{4 + 12x^4}{(x^4-1)^2}.$

Since $x \neq \pm 1$, $\dfrac{d^2y}{dx^2}$ is positive for all x in the domain.

15. a. $f(x) = \dfrac{2x+4}{x^2-k^2}$

$f'(x) = \dfrac{2(x^2-k^2) - (2x+4)(2x)}{(x^2-k^2)^2}$

$= -\dfrac{2x^2 + 8x + 2k^2}{(x^2-k^2)^2}$

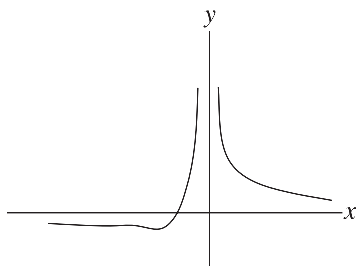
For critical values, $f'(x) = 0$ and $x \neq \pm k$:
$x^2 + 4x + k^2 = 0$

$x = \dfrac{-4 \pm \sqrt{16 - 4k^2}}{2}.$

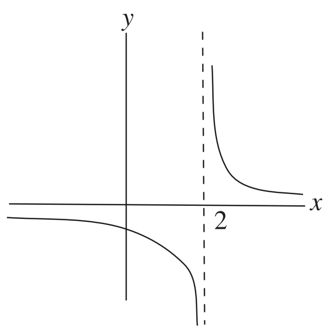
For real roots, $16 - 4k^2 \geq 0$
$-2 \leq k \leq 2.$
The conditions for critical points to exist are $-2 \leq k \leq 2$ and $x \neq \pm k$.

b. There are three different graphs that result for values of k chosen.

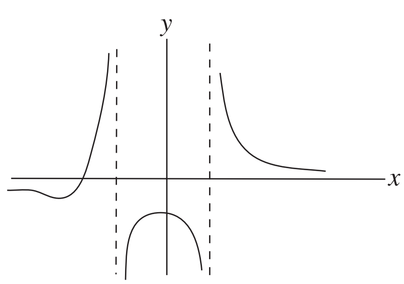
(i) $k = 0$



(ii) $k = 2$



(iii) For all other values of k, the graph will be similar to that of 1(*i*) in Exercise 9.5.



Chapter 9 Test

1. **a.** $x < -9$ or $-6 < x < -3$ or $0 < x < 4$ or $x > 8$

b. $-9 < x < -6$ or $-3 < x < 0$ or $4 < x < 8$

c. $(-9, 1)$, $(-6, -2)$, $(0, 1)$, $(8, -2)$

d. $x = -3$, $x = 4$

e. $f''(x) > 0$

f. $-3 < x < 0$ or $4 < x < 8$

g. $(-8, 0)$, $(10, -3)$

2. **a.** $g(x) = 2x^4 - 8x^3 - x^2 + 6x$

$g'(x) = 8x^3 - 24x^2 - 2x + 6$

To find the critical points, we solve $g'(x) = 0$:

$8x^3 - 24x^2 - 2x + 6 = 0$

$4x^3 - 12x^2 - x + 3 = 0$

Since $g'(3) = 0$, $(x - 3)$ is a factor.

$(x - 3)(4x^2 - 1) = 0$

$x = 3$ or $x = -\frac{1}{2}$ or $x = \frac{1}{2}$.

Note: We could also group to get $4x^2(x - 3) - (x - 3) = 0$.

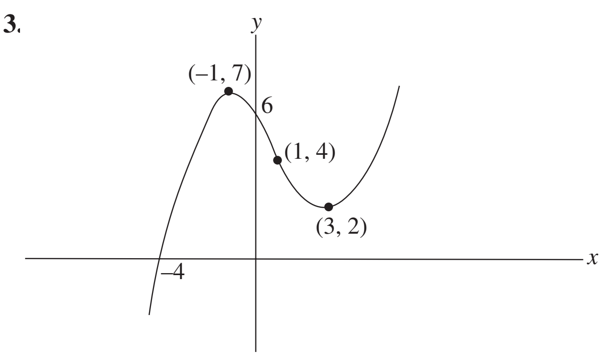
b. $g''(x) = 24x^2 - 48x - 2$

Since $g''\left(-\frac{1}{2}\right) = 28 > 0$, $\left(-\frac{1}{2}, -\frac{17}{8}\right)$ is a local maximum.

Since $g''\left(\frac{1}{2}\right) = -20 < 0$, $\left(\frac{1}{2}, \frac{15}{8}\right)$ is a local maximum.

Since $g''(3) = 70 > 0$, $(3, -45)$ is a local minimum.

3.



4. $g(x) = \dfrac{x^2 + 7x + 10}{(x - 3)(x + 2)}$

The function $g(x)$ is not defined at $x = -2$ or $x = 3$. At $x = -2$, the value of the numerator is 0. Thus, there is a discontinuity at $x = -2$, but $x = -2$ is not a vertical asymptote.

At $x = 3$, the value of the numerator is 40. $x = 3$ is a vertical asymptote.

$$g(x) = \frac{(x + 2)(x + 5)}{(x - 3)(x + 2)} = \frac{x + 5}{x - 3},\ x \neq -2$$

$$\lim_{x \to -2^-} g(x) = \lim_{x \to -2^-}\left(\frac{x + 5}{x - 3}\right) = -\frac{3}{5}$$

$$\lim_{x \to -2^+} g(x) = \lim_{x \to -2^+}\left(\frac{x + 5}{x - 3}\right) = -\frac{3}{5}$$

There is a hole in the graph of $g(x)$ at $\left(-2, -\frac{3}{5}\right)$.

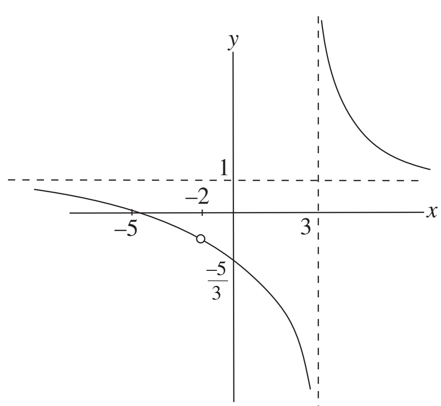
$$\lim_{x \to 3^-} g(x) = \lim_{x \to 3^-}\left(\frac{x + 5}{x - 3}\right) = -\infty$$

$$\lim_{x \to 3^+} g(x) = \lim_{x \to 3^+}\left(\frac{x + 5}{x - 3}\right) = \infty$$

There is a vertical asymptote at $x = 3$.

Also, $\lim\limits_{x \to \infty} g(x) = \lim\limits_{x \to -\infty} g(x) = 1$.

Thus, $y = 1$ is a horizontal asymptote.



5. $g(x) = e^{2x}(x^2 - 2)$

$g'(x) = e^{2x}(2)(x^2 - 2) + e^{2x}(2x)$

$\quad = 2e^{2x}(x^2 + x - 2)$

To find the critical points, we solve $g'(x) = 0$:

$2e^{2x}(x^2 + x - 2) = 0$

$(x + 2)(x - 1) = 0$, since $e^{2x} > 0$ for all x

$x = -2$ or $x = 1$.

Interval	$x < -2$	$x = -2$	$-2 < x < 1$	$x = 1$	$x > 1$
$g'(x)$	> 0	0	< 0	0	> 0
Graph of $g(x)$	Increas-ing	Local Max	Decreas-ing	Local Min	Increas-ing

The function $g(x)$ has a local maximum at $\left(-2, \frac{2}{e^4}\right)$ and a local minimum at $(1, -e^2)$.

6. $f(x) = \dfrac{2x + 10}{x^2 - 9}$

$\quad = \dfrac{2x + 10}{(x - 3)(x + 3)}$

There are discontinuities at $x = -3$ and at $x = 3$.

$$\left.\begin{aligned}\lim_{x \to -3^-} f(x) &= \infty\\ \lim_{x \to -3^+} f(x) &= -\infty\end{aligned}\right\}\ x = -3 \text{ is a vertical asymptote.}$$

$$\left.\begin{aligned}\lim_{x \to 3^-} f(x) &= -\infty\\ \lim_{x \to 3^+} f(x) &= \infty\end{aligned}\right\}\ x = 3 \text{ is a vertical asymptote.}$$

The y-intercept is $-\frac{10}{9}$ and $x = -5$ is an x-intercept.

$$f'(x) = \frac{2(x^2 - 9) - (2x + 10)(2x)}{(x^2 - 9)^2}$$

$$= \frac{-2x^2 - 20x - 18}{(x^2 - 9)^2}$$

For critical values, we solve $f'(x) = 0$:

$x^2 + 10x + 9 = 0$

$(x + 1)(x + 9) = 0$

$x = -1$ or $x = -9$.

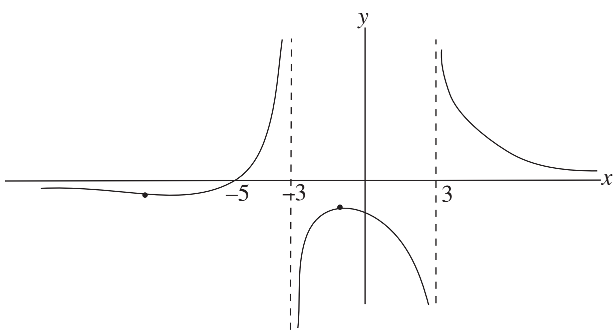
Interval	$x < -9$	$x = -9$	$-9 < x < -3$	$-3 < x < -1$	$x = -1$	$-1 < x < 3$	$x > 3$
$f'(x)$	< 0	0	> 0	> 0	0	< 0	< 0
Graph $f(x)$	Decreas-ing	Local Min	Increas-ing	Increas-ing	Local Max	Decreas-ing	Decreas-ing

$\left(-9, -\frac{1}{9}\right)$ is a local minimum and $(-1, -1)$ is a local maximum.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\frac{2}{x} + \frac{10}{x^2}}{1 - \frac{9}{x^2}} = 0 \text{ and}$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \left(\frac{\frac{2}{x} + \frac{10}{x^2}}{1 - \frac{9}{x^2}}\right) = 0$$

$y = 0$ is a horizontal asymptote.



7. $y = x^2 + \ln(kx)$

$= x^2 + \ln k + \ln x$

$$\frac{dy}{dx} = 2x + \frac{1}{x}$$

$$\frac{d^2y}{dx^2} = 2 - \frac{1}{x^2}$$

The second derivative is independent of k. There is not enough information to determine k.

8. $f(x) = x^3 + bx^2 + c$

a. $f'(x) = 3x^2 + 2bx$

Since $f'(-2) = 0$, $12 - 4b = 0$

$b = 3$.

Also, $f(-2) = 6$.

Thus, $-8 + 12 + c = 6$

$c = 2$.

b. $f'(x) = 3x^2 + 6x$

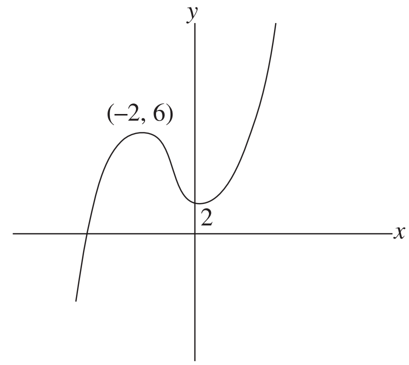
$= 3x(x + 2)$

The critical points are $(-2, 6)$ and $(0, 2)$.

$f''(x) = 6x + 6$

Since $f''(-2) = -6 < 0$, $(-2, 6)$ is a local maximum.

Since $f'(0) = 6 > 0$, $(0, 2)$ is a local minimum.



9. $y = x^{\frac{2}{3}}(x - 5)$

$= x^{\frac{5}{3}} - 5x^{\frac{2}{3}}$

$$\frac{dy}{dx} = \frac{5}{3}x^{\frac{2}{3}} - \frac{10}{3}x^{-\frac{1}{3}}$$

$$= \frac{5}{3}x^{-\frac{1}{3}}(x - 2)$$

$$= \frac{5(x-2)}{3x^{\frac{1}{3}}}$$

The critical values are $x = 2$ when $\frac{dy}{dx} = 0$,

and $x = 0$ when $\frac{dy}{dx}$ does not exist.

Interval	$x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$x > 2$
$\frac{dy}{dx}$	> 0	Does Not Exist	< 0	$= 0$	> 0
Graph of $y = f(x)$	Increas-ing	Local Max	Decreas-ing	Local Min	Increas-ing

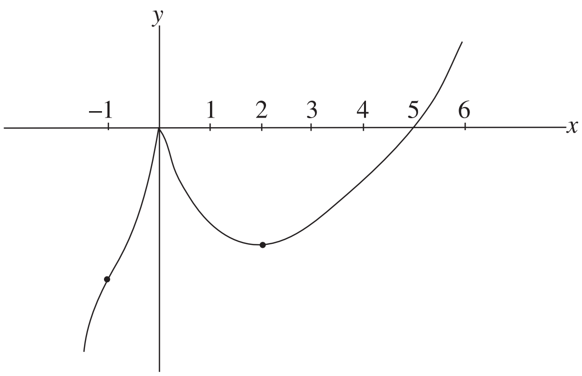
$$\frac{d^2y}{dx^2} = \frac{10}{9}x^{-\frac{1}{3}} + \frac{10}{9}x^{-\frac{4}{3}}$$

$$= \frac{10}{9}\left(\frac{x+1}{x^{\frac{4}{3}}}\right)$$

There are possible points of inflection at $x = -1$ and $x = 0$.

Interval	$x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$	$x > 0$
$\frac{d^2y}{dx^2}$	< 0	$= 0$	> 0	Does Not Exist	> 0
Graph of $y = f(x)$	Concave Down	Point of Inflection	Concave Up	Cusp	Concave Up

The y-intercept is 0. There are x-intercepts at 0 and 5.



10. $y = x^2 e^{kx} + p$

$$\frac{dy}{dx} = 2xe^{kx} + x^2(ke^{kx})$$

$$= xe^{kx}(2 + kx)$$

a. When $x = \frac{2}{3}$, $\frac{dy}{dx} = 0$.

Thus, $0 = \frac{2}{3}e^{\frac{2}{3}k}\left(2 + \frac{2}{3}k\right)$.

Since $e^{\frac{2}{3}k} > 0$, $2 + \frac{2}{3}k = 0$

$$k = -3.$$

b. The parameter p represents a vertical translation of the graph of $y = x^2e^{-3x}$.